Economics 142 Problem Set 2: Behavioral Game Theory Spring 2007
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This optional problem set covers the material in the second half of the course. The problems are meant to help you think about the issues we discuss in lectures, and also as practice for the final exam, which will be drawn partly from them. Before the final you are encouraged to work together and/or consult TA Rosalin Wu or me on the answers.

Unless otherwise noted below, all players are assumed to know the rules of the game they are playing, to know that all know the rules, and so on.

Iterated dominance and pure-strategy Nash equilibrium without dominance

1. In the game Pigs in a Box discussed in class (no animals were harmed in this question):

	Push	**Wait**
Push	**1** **5**	**5** **3**
Wait	**-1** **9**	**0** **0**

a. Find each pig's strictly dominated strategy or strategies, if any, and each pig's strategies that survive iterated deletion of strictly dominated strategies.

b. Find the Nash equilibrium or equilibria.

c. Justify your Nash equilibrium as the only possible outcome of the pigs' strategic thinking, making whatever assumptions about the pigs' rationality and/or knowledge of each other's rationality you need.

2. In the 3 × 3 game:

	L	C	R
T	0 7	5 0	3 0
M	0 5	2 2	0 5
B	7 0	5 0	3 7

a. Find each player's strictly dominated strategy or strategies, if any, and each player's strategies that survive iterated deletion of strictly dominated strategies.

b. Find the Nash equilibrium or equilibria.

c. Justify your Nash equilibrium or equilibria as the only possible outcome(s) of players' strategic thinking, making whatever assumptions about their rationality and/or knowledge of each other's rationality you need.

3. In the 2 × 3 game:

	L	C	R
T	10 4	0 3	3 1
B	0 0	10 2	3 10

a. Show that the game has no dominance by pure strategies for either player.

b. Show that R is strictly dominated for Column is we allow domination by mixed strategies, and identify a mixed strategy that dominates R.

c. Show that the game has a unique outcome that survives iterated deletion of strictly dominated strategies, and check that that outcome is the only pure-strategy Nash equilibrium. (It's also the only equilibrium in pure or mixed strategies, but you are not asked to show this.)

4. In the 3 × 3 game:

	L	C	R
T	0 7	5 0	7 0
M	0 5	2 2	0 5
B	7 0	5 0	0 7

a. Show that the game has no dominance by pure strategies for either player. (This game also has no dominance by mixed strategies for either player, but you are not asked to show this.)

b. Show that (M, C) is the unique pure-strategy Nash equilibrium. (This game also has no mixed-strategy Nash equilibria, but you are not asked to show this.)

c. (Optional) Show that the outcome (T, L) is nonetheless consistent with both players being rational, both knowing that each other is rational, both knowing that both know, and so on, by constructing a "tower" of rationally justified expectations (possibly different expectations for each player) in which each player is, or at least thinks s/he is, certain of the other's choice, and that support (T, L) as a rational choice for each player.

d. Which other pure-strategy outcomes in the game can you construct such a tower for?

e. Can you construct a tower to support an outcome other than (M, C) in which both players have the same expectations? Explain.

5. Find all pure-strategy Nash equilibria in the following game.

		Column		
		Left	Center	Right
Row	Top	1, -1	8, 6	3, -4
	Middle	-7, 100	6, 75	1, 200
	Bottom	5, 2	2, -1	0, 1

6. Consider the two-person game, in which both players know the value of x:

		Column L	Column R
Row	**U**	**1, 2**	**0, 1**
	D	**3, 0**	***x*, 1**

a. For what values of x (if any) is there a Nash equilibrium in which Column chooses R? Explain, and describe the equilibrium or equilibria.

b. For what values of x (if any) does strategy R for Column survive iterated deletion of strictly dominated strategies? Explain.

7. a. Modify the payoff matrix of the Battle of the Sexes game:

		Column (Woman) Fights	Ballet
Row (Man)	**Fights**	**1** **3**	**0** **0**
	Ballet	**0** **0**	**3** **1**

to reflect the assumption that—unlike in the original game—the man (Row player) prefers to be with the woman (Column player) at the ballet to being alone at the fights (and the man still prefers, as in the original game, being with the woman at the fights to being with the woman at the ballet); but also to reflect the assumption that—unlike in the original game—the woman prefers to be *without* the man, whether at the fights or at the ballet. (Choose whatever payoffs you like, as long as they have these features.)

b. Find both players' strictly dominated strategies, if any, and their strategies that survive iterated deletion of strictly dominated strategies.

c. Find the Nash equilibrium or equilibria.

8. Consider two-person guessing games as in the table below. In each game, each player has her/his own target, lower limit, and upper limit. Players make simultaneous guesses, which are required to be within their limits. Each player then earns $1000 minus the distance between her/his guess and the product of her/his target times the other's guess. These games are similar to Nagel's *n*–person guessing games. But unlike in Nagel's games, where the targets were usually fractions like ½ or 2/3 and the lower and upper limits were usually 0 and 100, here the targets can be greater than 1, the lower limits can be 200, 300, or 400, and the upper limits can be 500, 600, 700, or 900. Also unlike in Nagel's games where the targets and limits were always the same for both players, now the targets and/or the limits can be different for each player.

a.

	Lower Limit	Target	Upper Limit
Player 1	**200**	**0.7**	**600**
Player 2	**400**	**1.5**	**700**

b.

	Lower Limit	Target	Upper Limit
Player 1	**300**	**0.7**	**500**
Player 2	**400**	**1.3**	**900**

c.

	Lower Limit	Target	Upper Limit
Player 1	**400**	**0.5**	**900**
Player 2	**300**	**0.7**	**900**

d.

	Lower Limit	Target	Upper Limit
Player 1	**300**	**1.3**	**500**
Player 2	**200**	**1.5**	**900**

a-d. Find the Nash equilibrium or equilibria in games a-d.

e. (more difficult) In general, exactly how do the targets and limits determine the players' equilibrium guesses? Explain clearly, and prove your statement if you can.

f. Would you expect people randomly paired from students who have not studied game theory to play their equilibrium strategies in these guessing games? Explain why or why not. If not, explain what you think they might do instead.

Mixed-strategy and multiple equilibria

9. In the Alphonse and Gaston game:

		Go	Wait
		0	1
Go	**0**		**1**
		1	0
Wait	**1**		**0**

a. Find the mixed-strategy Nash equilibrium and explain why it is an equilibrium.

b. Compute players' equilibrium expected payoffs.

c. If the players cannot communicate or change the game, would you expect them to be able to coordinate on one of the more efficient pure-strategy equilibria? Why or why not?

10. In the Battle of the Sexes game:

			Column (Woman)	
			Fights	**Ballet**
			1	0
Row (Man)	**Fights**	**3**		**0**
			0	3
	Ballet	**0**		**1**

a. Find the mixed-strategy Nash equilibrium and explain why it is an equilibrium.

b. Compute players' equilibrium expected payoffs. Which would a player prefer, if he could choose: playing the mixed-strategy equilibrium or playing his least favorite pure-strategy equilibrium?

c. If the players cannot communicate or change the game, would you expect them to be able to coordinate on one of the more efficient pure-strategy equilibria? Why or why not?

11. Consider an Intersection game like Alphonse and Gaston, in which two drivers meet at the intersection of two roads, with one on each road and no way to distinguish between their roles. The payoffs are:

		Stop	**Go**
	Stop	**0, 0**	**1, 1**
	Go	**1, 1**	**0, 0**

a. Compute the mixed-strategy Nash equilibrium.

b. Would you expect this game to yield the drivers Pareto-efficient payoffs if they have no way to distinguish between their roles? Explain.

Now imagine that a stoplight is installed at the intersection, which both players can see before they decide whether to Go or Stop. The stoplight is Green for one driver if it is Red for the other, and is equally likely to be Green for Row and Red for Column or the reverse when they meet.

c. Show, in a new payoff matrix, how the stoplight changes the game and its set of equilibria. (Hint: The payoffs for the various outcomes are still as in the above matrix, but now players have more strategies because they can make their decision to Go or Stop depend on whether the light is Red or Green. To evaluate the consequences of their strategies, you will now have to make expected-payoff calculations that take into account their uncertainty about whether the light will be Green or Red for them and the effect this has on the final outcome.)

d. Would you expect this game to yield players Pareto-efficient payoffs if they have no way to distinguish between their roles? Explain.

12. Find a mixed-strategy Nash equilibrium in the following game. Find the players' equilibrium expected payoffs.

		Column		
		Left	**Center**	**Right**
	Top	**1, -1**	**8, 6**	**3, -4**
Row	**Middle**	**-7, 100**	**6, 75**	**1, 200**
	Bottom	**5, 2**	**2, -1**	**0, 1**

13. Consider a Matching Pennies game but with the payoff to the Row player (and from the Column player) for matching on Heads raised from 1 to 2. (The payoff to the Row player and from the Column player for matching on Tails remains 1).
a. Write the payoff matrix of this version of Matching Pennies.

b. Find the Nash equilibrium and players' equilibrium expected payoffs in this game.

c. Compared with the original Matching Pennies game, in which direction does Row's equilibrium probability of playing Heads change? Does this direction correspond to your intuition about the effect of increasing the payoff to matching on Heads? Explain.

d. In which direction does Column's equilibrium probability of playing Heads change? Does this direction correspond to your intuition about the effect of decreasing the payoff to matching on Heads? Explain.

e. In which direction do Row's and Column's equilibrium expected payoffs change?

14. Consider a Chicken-like game where two players are driving toward each other on a one-lane road with shoulders. Each player chooses between going straight, swerving to the left, and swerving to the right. The choices are made simultaneously. If one goes straight while the other swerves right or left, the one who goes straight gets a payoff of 3 while the other gets –1. If each swerves to his left, each gets 0. If each swerves to his right, again each gets 0. If both go straight, or if one swerves to his left while the other swerves to his right, then the cars crash and each gets the payoff –4.

a. Write the payoff matrix for this game.

b. Find all Nash equilibria in pure strategies.

c. Find the Nash equilibrium where each player uses all three of his strategies in a mixture. (Hint: First show that the probabilities of Left and Right must be equal in equilibrium for each player. Then equate the expected payoffs of Left, Straight, and Right and use your equations to calculate their probabilities of Straight.)

15. a. Find all pure- or mixed-strategy Nash equilibria for the two-person Stag Hunt game:

		Other Player: Stag	Other Player: Rabbit
Self	Stag	2, 2	1, 0
	Rabbit	0, 1	1, 1

b. What strategy would you play, against one randomly selected UCSD student (not a member of this class), in the two-person Stag Hunt game? Explain.

c. Find all pure- or mixed-strategy equilibria for the *n*-person Stag Hunt game:

		All Other Players: All-Stag	All Other Players: Not All-Stag
Self	Stag	2	0
	Rabbit	1	1

d. What strategy would you play, against one randomly selected UCSD student (not a member of this class), in a 10-person Stag Hunt game? Explain.

16. (“Nash’s demand game”) Two risk-neutral, expected money-maximizing bargainers, U and V, must agree on how to share $1. They bargain by making simultaneous demands; if their demands add up to more than $1, they each get nothing; if they add up to less than or equal to $1, each bargainer gets exactly his demand. Assume that any real number is a possible demand, and is also a possible division of the money.

a. Find an infinite number of pure-strategy Nash equilibria in this game. (Hint: Pick a demand for one player and try to find a demand for the other player that is in equilibrium with it. Then do it again, and again.) Explain why, in one of your equilibria, neither bargainer can do better by switching to another strategy.

b. Find an infinite number of mixed-strategy Nash equilibria in this game. (Hint: Pick two demands for one player and try to find demands for the other player and mixing probabilities for both players that are in equilibrium with your initial demands. Then do it again, and again.) Explain why, in one of your equilibria, neither bargainer can do better by switching to another strategy, pure or mixed.

c. Show how to compute the probability of disagreement in one of your mixed strategy equilibria, and show that it is always strictly positive.

d. Are there any Pareto-efficient equilibria in this game?

e. Now suppose that there are two plausible, but different, notions of what it means to divide the dollar fairly. Redo your analysis from part b, assuming that bargainers can put positive probability only on demands that are consistent with one or the other notion of fairness. Is the equilibrium identified here also an equilibrium in the original game?

f. Give a real-world (not experimental) example in which common ideas of fairness appear to determine bargaining outcomes as in your answer to e.

17. Consider a game with payoff matrix as shown, with no communication. The game is presented to players as a story, without a matrix: “Each player chooses either X or Y. If you both choose X then you each get $5. If you both choose Y then you each get $5. If you choose differently, then neither one of you receives any money.” It is known that both players are told the same story, so the common labeling of the actions is known.

	X	Y
X	**5** **5**	**0** **0**
Y	**0** **0**	**5** **5**

a. Find the pure-strategy and mixed-strategy equilibria.

b. What strategy would you play, first as row and then as column player, against one randomly selected UCSD student (not a member of this class), in this game? Would your strategy (X or Y) differ for the row or the column player? Explain.

Now suppose the game is exactly as before, except that payoffs are changed as follows:

	X	Y
X	**5.1** **5**	**0** **0**
Y	**0** **0**	**5** **5.1**

Note that the 5.1 payoff for the row player is one of the possible outcomes if he plays Y, but the 5.1 payoff for the column player is one of the possible outcomes if he plays X. Thus the payoffs interact differently with the labels for the row and the column player.

c. Find the pure-strategy and mixed-strategy equilibria.

d. What strategy would you play, as row player, against one randomly selected UCSD student (not a member of this class)? Explain.

e. What strategy would you play, as column player, against one randomly selected UCSD student (not a member of this class)? Would your strategy (X or Y) differ for the row or the column player? Explain.

18. Suppose three identical, risk-neutral firms must decide simultaneously whether to enter a new market which can accommodate only two of them. If all three firms enter, all get payoff 0; otherwise, entrants get 9 and firms that stay out get 8.

a. Identify the unique mixed-strategy equilibrium and describe the resulting probability distribution of the total ex post number of entrants. (You are not asked to show this, but the game also has three pure-strategy equilibria, in each of which exactly two firms enter; but these equilibria are arguably unattainable in a one-shot game in the absence of prior agreement or precedent. The mixed-strategy equilibrium is symmetric, hence attainable.)

Now suppose that each firm follows a behavioral rule that is an independent and identically distributed draw from a distribution that assigns equal probabilities to two "level-*k*" types: either *L1* (best response assuming the other firms are each equally likely to enter or stay out, and probabilistically independent), or *L2* (best response to *L1*).

b. Describe the decisions of types *L1* and *L2* and the resulting *actual* (as opposed to what *L1* or *L2* expect) probability distribution of the total ex post number of entrants when each firm's type is drawn as explained above. Show that the expected number of entrants is closer to the ex post optimal number (2) than in your equilibrium from part (a), and that that the probability of exactly 2 entrants is higher than in (a). (In experiments subjects' initial responses come systematically closer to ex post optimality than the symmetric mixed-strategy equilibrium predicts, a result Kahneman has described as "magic." This analysis shows that bounded strategic rationality works like fairy dust.)

Backward induction and subgame-perfectness

19. Reconsider the Battle of the Sexes game:

		Column (Woman): Fights	Column (Woman): Ballet
Row (Man)	Fights	3, 1	0, 0
	Ballet	0, 0	1, 3

a. Find the subgame-perfect equilibrium in the game in which Column can choose her strategy first and Row can observe it before choosing his strategy. Would you expect the players to be able to coordinate on one of the (more efficient) pure-strategy equilibria in this game? Why or why not?

b. Show that in the subgame-perfect equilibrium in a, even the Row player is better off than in the mixed-strategy equilibrium of the original game in problem 10.

c. What would you expect to happen if Column chooses her strategy first and Row does NOT get to observe her choice before choosing his strategy?

d. What would you expect to happen if, in the original game, Column could make a nonbinding suggestion about strategies to Row before players choose their decisions?

e. (Optional) What would you expect to happen if, in the original game, both Row and Column could make simultaneous, nonbinding suggestions about strategies before players choose their decisions?

20. Consider a two-person game with payoff matrix as shown. Assume that both players know the rules of the game, including the values of *x* and y, and know they know, etc.

	L	R
T	2 2	x 0
B	1 1	y 1

a. For what values of x and y are T for Row and L for Column in equilibrium?

Now suppose that before Row and Column play the above game, Column must send R a costless, nonbinding message announcing her/his intention to play either L or R. (Because the message is costless and nonbinding, it has no direct effect on players' payoffs, but there could be equilibria or subgame-perfect equilibria in which players use strategies in which their decisions depend on Column's message.)

b. For what values of x and y are T for Row and L for Column in subgame-perfect equilibrium in the entire game, including the announcement?

21. Two players must agree how to divide a prize. Baker decides how large the total prize will be, either $10 or $100 (the only choices). Cutler simultaneously chooses how to divide the prize chosen by Baker, either a 50%-50% split or a 90%-10% split (the only choices), where in the latter case Cutler gets 90%. The payoffs, in dollars, are:

		Baker	
		$10 prize	**$100 prize**
Cutler	**50:50**	**5, 5**	**50, 50**
	90:10	**9, 1**	**90, 10**

First suppose that the players must choose simultaneously, and cannot change the game.

a. Find the Nash equilibrium or equilibria.

Now suppose that the basic choices are the same, but that one or more *strategic moves* may be possible. These could involve one player or the other observably committing to his choice before the other player chooses, or observably committing to follow a contingent rule that makes his or her choice, now made after the other player chooses, depend on the other player's choice in some way.

b. Which, if either, player could benefit from making a strategic move in this game, and by which kind of strategic move? Explain how the strategic move works and its effect on the outcome, using the idea of rollback (subgame-perfect) equilibrium. (Hint: Consider contingent threats of the form: "I will choose *X* if you choose *Y*.")

22. Two people, Rhoda ("R") and Colin ("C") must decide independently whether to try to meet at the fights ("F") or the ballet ("B"). Before R and C decide where to try to meet, R (but not C) must send a message about her intentions, "f" for F or "b" for B, about where she plans to go. Then R and C choose, simultaneously and independently, between going to F and going to B. The message is nonbinding, so that R is free to choose either F or B independent of her announcement f or b; but if R announces f but chooses B, or announces b but chooses F, she incurs a cost $c \geq 0$. If R and C both choose F, R's payoff is 2 less whatever cost c she incurs, and C's payoff is 1. If R and C both choose B, R's payoff is 1 less whatever cost c she incurs, and C's payoff is 2. If R chooses F but C chooses B, or vice versa, R's payoff is 0 less whatever cost c she incurs, and C's payoff is 0. For example, if R announces f, chooses B, and C chooses B, R's payoff is $1 - c$.

a. Clearly identifying players' decisions and information sets, draw the game tree. (It's easier to draw the information sets if you put both of R's decisions first.)

b. Identify the pure-strategy subgame-perfect equilibrium outcome(s) and payoffs when $c > 2$.

c. Identify the pure-strategy subgame-perfect equilibrium outcome(s) and payoffs when $0 \leq c < 1$.

Learning and equilibrium selection

23. Show or argue (informally or graphically, if you prefer) that if players repeatedly play the game Pigs in a Box discussed in class, then increasing the probabilities of playing their strategies that do better over time and decreasing the probabilities of playing their strategies that do worse, they will converge to the equilibrium outcome (Push, Wait).

	Push	**Wait**
Push	**1** **5**	**5** **3**
Wait	**-1** **9**	**0** **0**

24. In the following environments, a large number of identical players choose simultaneously between two pure strategies. In each case, graph the payoffs of the two strategies against the population frequency of the first-mentioned strategy in a way that is consistent with the verbal description. Then use your graph to determine what pattern (or patterns) of behavior will emerge in the long run, and whether those pattern(s) will be Pareto-efficient, in the sense of maximizing all players' average expected payoff.

a. Each person can either install a car alarm in his car or not. Car alarms are highly effective when only a few cars have them, but (because people ignore them when they go off too often) they are ineffective (but still costly) when most cars have them.

b. There is a wall running through the center of your city, left over from the Cold War. Each person can either try to tear down the wall or ignore it. Everyone hates the wall, but everyone knows that if only a few people try to tear it down the government will arrest them and send them to jail. However, everyone also knows that if more than a few people try to tear it down, the government is unlikely to punish them.

c. Each person can either shirk or work hard. Working hard has higher cost, other things equal, but each person also benefits by a constant amount, greater than the cost difference between working hard and shirking, from having an effort level the same as the most common effort level in the population. (Ignore the benefits of working hard.)

25. Suppose that the speed limit is 70 on the freeway, and that a large number N of drivers simultaneously and independently choose speeds from 70 to 100. Everyone prefers to go as fast as possible, other things equal, but the police are sure to ticket any driver whose speed is strictly faster than x% of the drivers, where x is a given parameter such that $0 < x < 100$. (Thus, only by driving exactly 70 can a driver be sure of not being ticketed.) Suppose further that each driver ignores his own influence on the percentage, and that the cost of being ticketed outweighs any benefit of going faster.

a. Model this situation as a noncooperative game and analyze its set of pure-strategy Nash equilibria as far as possible. (Assume that x is a multiple of $1/N$, that is $x = k/N$, where k is an integer.)

b. Does the set of Nash equilibria depend on x when $0 < x < 100$?

c. What is the set of pure-strategy Nash equilibria when the police don't ticket anyone? Explain.

d. What is the set of pure-strategy Nash equilibria when the police ticket everyone who speeds? Explain.

e. If the same drivers play this game repeatedly, observing the outcome after each play, how would you expect their speeds to change over time as they learn to predict each other's speeds? Explain intuitively or formally, whichever you prefer.

26. Reconsider the Intersection game from problem 11, in which two drivers meet at the intersection of two roads, with one on each road, and with no way to distinguish their roles. Suppose that many drivers, in a large population, repeatedly, randomly meet in pairs to play the Intersection game.

	Stop	**Go**
Stop	**0, 0**	**1, 1**
Go	**1, 1**	**0, 0**

a. What aggregate pattern or patterns (population frequencies) of Stop and Go decisions would you expect to see emerge in this population in the long run? Explain.

Now assume that a stoplight is installed at the intersection, which both players can see before they decide whether to Go or Stop. The stoplight is Green for one driver if it is Red for the other, and is equally likely to be Green for Row and Red for Column or the reverse when they meet.

b. Using your answer to problem 11 (showing how the stoplight changes the game and its set of equilibria), what aggregate pattern or patterns (that is, what population frequencies) of Stop and Go decisions would you expect to see emerge in this population in the long run? Explain.

27. Reconsider the game Pigs in a Box from problem 1:

	Push	**Wait**
Push	**1** **5**	**5** **3**
Wait	**-1** **9**	**0** **0**

Answer the following questions, assuming that players are repeatedly paired at random from a large population to play this game, and that they adjust their strategies over time in a way that always increases the population frequency of a pure strategy that has higher expected payoff, given the current mix of strategies in the population.

a. Using your answer to problem 1, show graphically or algebraically that the population will converge to a state in which Column players all play Wait.

b. Using your answer to problem 1, show graphically or algebraically or intuitively that the population will eventually converge to a state in which Row players all play Push.