

Online Technical Appendix to
**The Extensive Margin of Exporting Products:
A Firm-level Analysis***

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August 3, 2011

Abstract

We generalize the Arkolakis and Muendler (2010a) model to nested consumer preferences. Each inner nest holds the products in a firm's product line with an elasticity of substitution that differs from that of the outer nest over product lines of different firms. We find that all main results for a firm's product choice, the sales distribution within and across firms, and the aggregation results also hold in this generalized setup, compared to the simple case where the elasticities are the same between products in a product line and between product lines.

Keywords: International trade; heterogeneous firms; multi-product firms; firm and product panel data; Brazil

JEL Classification: F12, L11, F14

*We thank Elhanan Helpman and Sam Kortum for valuable comments on this note. Any remaining errors are ours. We acknowledge NSF support (SES-0550699 and SES-0921673) with gratitude.

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1 Nested Preferences with Different Elasticities

We analyze a generalized version of the Arkolakis and Muendler (2010a) model of multi-product firms that allows for within-firm cannibalization effects. The main result is that the qualitative properties of the Arkolakis and Muendler (2010a) model are retained: the size distribution of firm sales and the distribution of the firms' numbers of products is consistent with regularities in Brazilian exporter data as well as other data sets. More importantly, the general equilibrium properties of the model do not depend on the lower tier elasticity (the elasticity across the products of a given firm) so that the general equilibrium of the model can be easily characterized using the tools of Dekle, Eaton, and Kortum (2007).

While the model is highly tractable, the introduction of one more demand elasticity adds a further degree of freedom. This degree of freedom can be disciplined using independent estimates for the upper and lower tier elasticities, such as those of Broda and Weinstein (2006). Under an according parametrization, the model can be used for counterfactual exercises that simulate the impact of changes in trade costs on the firm size distribution and the distribution of the firms' numbers of products.

In the next section we present and solve for the generalized model. Section 3 concludes.

2 Model

There is a countable number of countries. We label the source country of an export shipment with s and the export destination with d .

We adopt a two-tier CES utility function for consumer preferences.¹ Each inner nest of consumer preferences aggregates a firm's products with a CES utility function and an elasticity of substitution ε . Using marketing terminology, the product bundle of the inner nest can be called a firm's product line (or product mix). The product lines of different firms are then aggregated using an outer CES utility nest with an elasticity σ . Each firm offers a countable number of products but there is a continuum of firms in the world. We assume that every product line is uniquely offered

¹Atkeson and Burstein (2008) use a similar nested CES form in a heterogeneous-firms model of trade but their upper tier refers to different industries and the lower tier to different firms within the industry. Eaton and Kortum (2010) present a stochastic model with nested CES preferences to characterize the firm size distribution and their products under Cournot competition. In our model, firms do not strategically interact with other firms. This property of the model allows us to characterize general equilibrium beyond the behavior of individual firms.

by a single firm, but a firm may ship different product lines to different destinations. Formally, the representative consumer's utility function at destination d is given by

$$U_d = \left(\sum_s \int_{\Omega_{sd}} \left(\sum_{g=1}^{G_{sd}(\omega)} q_{sdg}(\omega)^{\frac{\varepsilon-1}{\varepsilon}} \right)^{\frac{\sigma-1}{\sigma} \frac{\varepsilon}{\varepsilon-1}} d\omega \right)^{\frac{\sigma}{\sigma-1}}$$

where $q_{sdg}(\omega)$ is the quantity consumed of the g -th product of firm ω , producing in country s . Ω_{sd} is the set of firms from source country s selling to country d .

The representative consumer's first order conditions imply that demand for the g -th product of firm ω in market d is

$$q_{sdg}(\omega) = p_{sdg}(\omega)^{-\varepsilon} P_{sd}(\omega; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d,$$

where $p_{sdg}(\omega)$ is the price of that product,

$$P_{sd}(\omega; G_{sd}) \equiv \left[\sum_{g=1}^{G_{sd}(\omega)} p_{sdg}(\omega)^{-(\varepsilon-1)} \right]^{-1/(\varepsilon-1)}$$

is the ideal price index for the product line of firm ω selling $G_{sd}(\omega)$ products in market d , and

$$P_d \equiv \left[\sum_s \int_{\Omega_{sd}} P_{sd}(\omega; G_{sd})^{-(\sigma-1)} d\omega \right]^{-1/(\sigma-1)}$$

is the ideal consumer price index in market d . T_d is total consumption expenditure.

2.1 Firm Optimization

We assume that the firm has a linear production function for each product. A firm with overall productivity ϕ faces an efficiency $\phi/h(g)$ in producing its g 'th product, where $h(g)$ is an increasing function with $h(1) \equiv 1$. We call the firm's total number of products G_{sd} at destination d its *exporter scope* at d . Productivity is the only source of firm heterogeneity so that, under the model assumptions below, firms of the same type ϕ from country s face an identical optimization problem in every destination d . Since all firms with productivity ϕ will make identical decisions in equilibrium, it is convenient to name them by their common characteristic ϕ from now on.

The firm also incurs local entry costs to sell its g -th product in market d : $f_{sd}(g) > 0$ for $g > 1$, with $f_{sd}(0) = 0$. These incremental product-specific fixed costs may increase or decrease with

exporter scope. The overall entry cost for market d is denoted by $F_{sd}(G) \equiv \sum_{g=1}^G f_{sd}(g)$ and strictly increases in exporter scope by definition.

Profits of a firm with productivity ϕ from country s that sells products $g = 1, \dots, G_{sd}$ in d at prices p_{sdg} are

$$\pi_{sd}(\phi) = \sum_{g=1}^{G_{sd}} \left(p_{sdg} - \frac{w_s}{\phi/h(g)} \tau_{sd} \right) p_{sdg}^{-\varepsilon} \cdot P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d - F_{sd}(G_{sd}). \quad (1)$$

We consider the first-order conditions with respect to the prices p_{sdg} of each product g , consistent with an optimal product-line price $P_{sd}(\phi; G_{sd})$, and also with respect to exporter scope G_{sd} . As shown in Appendix A, the first-order conditions with respect to prices imply a constant markup over marginal cost for all products equal to $\tilde{\sigma} \equiv \sigma / (\sigma - 1)$.

Using the constant markup rule in demand for the g -th product of a firm with exporter scope G_{sd} yields optimal sales of the product

$$p_{sdg}(\phi) q_{sdg}(\phi) = \left(\tilde{\sigma} \frac{w_s \tau_{sd}}{\phi/h(g)} \right)^{-(\varepsilon-1)} P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d. \quad (2)$$

Using this result and the definition of $P_{sd}(\phi; G_{sd})$, we can write profits for a firm selling G_{sd} products as

$$\begin{aligned} \pi_{sd}(\phi; G_{sd}) &= P_{sd}(\phi; G_{sd})^{-(\sigma-1)} \frac{P_d^{\sigma-1} T_d}{\sigma} - F_{sd}(G_{sd}) \\ &= H(G_{sd})^{-(\sigma-1)} (\tilde{\sigma} w_s \tau_{sd})^{-(\sigma-1)} \frac{\phi^{\sigma-1} P_d^{\sigma-1} T_d}{\sigma} - F_{sd}(G_{sd}), \end{aligned} \quad (3)$$

where

$$H(G_{sd}) \equiv \left[\sum_{g=1}^{G_{sd}} h(g)^{-(\varepsilon-1)} \right]^{-1/(\varepsilon-1)}$$

is the firm's product efficiency index.

We impose the following assumption, which is necessary for optimal exporter scope to be well defined.

Assumption 1 *Parameters are such that $Z_{sd}(G) = f_{sd}(G)/[H(G)^{-(\sigma-1)} - H(G-1)^{-(\sigma-1)}]$ strictly increases in G .*

The expression for $Z_{sd}(G)$ reduces to $Z_{sd}(G) = f_{sd}(G)h(G)^{\sigma-1}$ when $\varepsilon = \sigma$. So, in that case, Assumption 1 is identical to the one considered in Arkolakis and Muendler (2010a).

For a firm to enter a destination market, its productivity has to exceed a threshold ϕ_{sd}^* , where ϕ_{sd}^* is implicitly defined by zero profits for the first product:

$$\frac{1}{\sigma} P_d^{\sigma-1} T_d [P_{sd}(\phi_{sd}^*; 1)]^{-(\sigma-1)} = f_{sd}(1).$$

Using the convention $h(1) = 1$ for $G = 1$ in (3) yields

$$(\phi_{sd}^*)^{\sigma-1} = \sigma f_{sd}(1) \frac{(\tilde{\sigma} w_s \tau_{sd})^{\sigma-1}}{P_d^{\sigma-1} T_d}. \quad (4)$$

Similarly, we can define the threshold productivity of selling G products in market d . The firm is indifferent between introducing a G -th product or stopping with an exporter scope of $G - 1$ at the product-entry threshold $\phi_{sd}^{*,G}$ if

$$\pi_{sd}(\phi_{sd}^{*,G}; G) - \pi_{sd}(\phi_{sd}^{*,G}; G - 1) = 0. \quad (5)$$

Using equations (3) and (4) in this profit equivalence condition, we can solve out for the implicitly defined product-entry threshold $\phi_{sd}^{*,G}$, at which the firm sells G_{sd} or more products,

$$\left(\phi_{sd}^{*,G}\right)^{\sigma-1} = \frac{(\phi_{sd}^*)^{\sigma-1}}{H(G_{sd})^{-(\sigma-1)} - H(G_{sd} - 1)^{-(\sigma-1)}} \frac{f_{sd}(G_{sd})}{f_{sd}(1)} = \frac{(\phi_{sd}^*)^{\sigma-1}}{f_{sd}(1)} Z_{sd}(G_{sd}), \quad (6)$$

where we define $\phi_{sd}^{*,1} \equiv \phi_{sd}^*$. So, under Assumption 1, the profit equivalence condition (5) implies that the product-entry thresholds $\phi_{sd}^{*,G}$ strictly increase with G and more productive firms will weakly raise exporter scope compared to less productive firms. Based on these relationships, we can make the following statement

Export sales can be written succinctly using equation (4)

$$\begin{aligned} t_{sd}(\phi) &= \left(\tilde{\sigma} \frac{w_s \tau_{sd}}{\phi}\right)^{1-\varepsilon} P_d^{\sigma-1} T_d \sum_{g=1}^{G_{sd}} h(g)^{1-\varepsilon} [P_d(G_{sd}(\phi))]^{\varepsilon-\sigma} \\ &= \sigma f_{sd}(1) \left(\frac{\phi}{\phi_{sd}^*}\right)^{\sigma-1} H(G_{sd}(\phi))^{-(\sigma-1)} \end{aligned} \quad (7)$$

Notice that the relationship is similar in models with $\varepsilon \neq \sigma$ or $\varepsilon = \sigma$. The only difference is that $H(G_{sd})$ changes in the two cases since ε can take different values. Note that, if the term $H(G_{sd})$ converges to a constant for $G_{sd} \rightarrow \infty$, then export sales are Pareto distributed in the upper tail if ϕ is Pareto distributed.

Proposition 1 *Suppose Assumption 1 holds. Then for all s, d :*

- *exporter scope* $G_{sd}(\phi)$ is positive and weakly increases in ϕ for $\phi \geq \phi_{sd}^*$;
- *total firm exports* $t_{sd}(\phi)$ are positive and strictly increase in ϕ for $\phi \geq \phi_{sd}^*$.

Proof. The first statement follows directly from the discussion above. The second statement follows because $H(G_{sd}(\phi))^{-(\sigma-1)}$ strictly increases in $G_{sd}(\phi)$ and $G_{sd}(\phi)$ weakly increases in ϕ so that $t_{sd}(\phi)$ strictly increases in ϕ by (7). ■

We can also write exporter mean sales in market d as

$$a_{sd}(\phi) = \sigma f_{sd}(1) \left(\frac{\phi}{\phi_{sd}^*} \right)^{\sigma-1} \frac{H(G_{sd}(\phi))^{1-\sigma}}{G_{sd}(\phi)}$$

Under a mild condition, sales per export product $a_{sd}(\phi)$ increase in ϕ and thus with firm total sales. For this to be the case the following is a sufficient condition.

Case 1 *The function $Z_{sd}(g)$ strictly increases in g with an elasticity*

$$\frac{\partial \ln Z_{sd}(g)}{\partial \ln g} > 1.$$

Case 1 is more restrictive than Assumption 1 in that the condition not only requires Z_{sd} to increase with g but that the increase be more than proportional. We can now formally state

Proposition 2 *If $Z_{sd}(g)$ satisfies Case 1, then sales per export product $a_{sd}(\phi)$ strictly increase at the discrete points $\phi = \phi_{sd}^*, \phi_{sd}^{*,2}, \phi_{sd}^{*,3}, \dots$*

Proof. Although $Z_{sd}(g)$ is defined in more general terms now, it enters the relevant relationships in the same way as in Arkolakis and Muendler (2010a) before. So, Case 1 applies in the same way. See the Appendix in Arkolakis and Muendler (2010a) for details of the proof. ■

2.2 Within-firm Sales Distribution

We revisit optimal sales per product and their relationship to exporter scope and the product's rank in a firm's sales distribution. The relationship lends itself to estimation in micro data. Using the productivity thresholds for firm entry (4) and product entry (6) in optimal sales (2) and simplifying yields

$$\begin{aligned} p_{sdg}(\phi)x_{sdg}(\phi) &= \sigma Z_{sd}(G_{sd})H(G_{sd})^{\varepsilon-\sigma} \left(\frac{\phi}{\phi_{sd}^{*,G}} \right)^{\sigma-1} h(g)^{-(\varepsilon-1)} \\ &= \sigma \frac{f_{sd}(G_{sd})H(G_{sd})^{\varepsilon-1}}{1 - [1 - h(G_{sd})^{-(\varepsilon-1)}/H(G_{sd})^{-(\varepsilon-1)}]^{\frac{\sigma-1}{\varepsilon-1}}} \left(\frac{\phi}{\phi_{sd}^{*,G}} \right)^{\sigma-1} h(g)^{-(\varepsilon-1)}. \quad (8) \end{aligned}$$

Note that $H(G)^{\varepsilon-\sigma}$ strictly falls in G if $\varepsilon > \sigma$. Under Case 1, the term $Z_{sd}(G_{sd})H(G_{sd})^{\varepsilon-\sigma}$ must strictly increase in G , however, because individual product sales strictly drop as scope g increases and $h(g)^{-(\varepsilon-1)}$ falls. So, if $Z_{sd}(G_{sd})H(G_{sd})^{\varepsilon-\sigma}$ did not strictly increase in G , average sales per product would not strictly increase, contrary to Proposition 2.

Compared to Arkolakis and Muendler (2010a), the relationship (8) is not log-linear if $\varepsilon \neq \sigma$ and requires a non-linear estimator, similar to the general case in continuous product space (Arkolakis and Muendler 2010b).

2.3 Aggregation

To derive clear predictions for the model equilibrium we specify a Pareto distribution of firm productivity following Helpman, Melitz, and Yeaple (2004) and Chaney (2008). A firm's productivity ϕ is drawn from a Pareto distribution with a source-country dependent location parameter b_s and a shape parameter θ over the support $[b_s, +\infty)$ for all destinations s . So the cumulative distribution function of ϕ is $\Pr = 1 - (b_s)^\theta / \phi^\theta$ and the probability density function is $\theta(b_s)^\theta / \phi^{\theta+1}$, where more advanced countries are thought to have a higher location parameter b_s . Therefore the measure of firms selling to country d , that is the measure of firms with productivity above the threshold ϕ_{sd}^* , is

$$M_{sd} = J_s \frac{b_s^\theta}{(\phi_{sd}^*)^\theta}. \quad (9)$$

Furthermore, the probability density function of the conditional productivity distribution for entrants is given by

$$\mu_{sd}(\phi) = \begin{cases} \theta(\phi_{sd}^*)^\theta / \phi^{\theta+1} & \text{if } \phi \geq \phi_{sd}^* \\ 0 & \text{otherwise.} \end{cases} \quad (10)$$

We define the resulting Pareto shape parameter of the total sales distribution as $\tilde{\theta} \equiv \theta / (\sigma - 1)$.

With these distributional assumptions we can compute a number of aggregate statistics from the model. We denote aggregate bilateral sales of firms from s to country d as T_{sd} . The corresponding average sales are defined as \bar{T}_{sd} , so that $T_{sd} = M_{sd}\bar{T}_{sd}$ and

$$\bar{T}_{sd} = \int_{\phi_{sd}^*}^{\infty} t_{sd}(\phi) \mu_{sd}(\phi) d\phi. \quad (11)$$

Similarly, we define average local entry costs as

$$\bar{F}_{sd} = \int_{\phi_{sd}^*}^{\infty} F_{sd}(G_{sd}(\phi)) \mu_{sd}(\phi) d\phi.$$

To compute \bar{T}_{sd} , we impose two additional assumptions.

Assumption 2 Parameters are such that $\theta > \sigma - 1$.

Assumption 3 Parameters are such that $\tilde{F}_{sd} \equiv \sum_{g=1}^{\infty} f_{sd}(g)^{1-\tilde{\theta}} [H(g)^{1-\sigma} - H(g-1)^{1-\sigma}]^{\tilde{\theta}}$ is strictly positive and finite.

Then we can make the following statement.

Proposition 3 Suppose Assumptions 1, 2 and 3 hold. Then average sales \bar{T}_{sd} per firm are a constant multiple of average local entry costs \bar{F}_{sd}

$$\bar{T}_{sd} = \frac{\tilde{\theta}\sigma}{\tilde{\theta} - 1} \bar{F}_{sd} = f_{sd}(1)^{\tilde{\theta}} \tilde{F}_{sd}.$$

Proof. See Appendix C. ■

As a result, bilateral expenditure trade shares can be expressed as

$$\lambda_{sd} = \frac{M_{sd} \bar{T}_{sd}}{\sum_k M_{kd} \bar{T}_{kd}} = \frac{J_s(b_s)^\theta (w_s \tau_{sd})^{-\theta} f_{sd}(1)^{-\tilde{\theta}} \bar{F}_{sd}}{\sum_k J_k(b_k)^\theta (w_k \tau_{kd})^{-\theta} f_{kd}(1)^{-\tilde{\theta}} \bar{F}_{kd}}, \quad (12)$$

an expression that depends on the values of ε and σ only insofar as these parameters affect \bar{F}_{sd} through $H(g)$.

We can also compute mean exporter scope in a destination. This number is given by

$$\begin{aligned} \bar{G}_{sd} &= \int_{\phi_{sd}^*} G_{sd}(\phi) \mu_{sd}(\phi) d\phi \\ &= (\phi_{sd}^*)^\theta \theta \left[\int_{\phi_{sd}^*}^{\phi_{sd}^{*,2}} \phi^{-\theta-1} d\phi + \int_{\phi_{sd}^{*,2}}^{\phi_{sd}^{*,3}} 2\phi^{-\theta-1} d\phi + \dots \right] \\ &= \frac{(\phi_{sd}^{*,2})^{-\theta} - (\phi_{sd}^*)^{-\theta}}{(\phi_{sd}^*)^{-\theta}} + \frac{(\phi_{sd}^{*,3})^{-\theta} - (\phi_{sd}^{*,2})^{-\theta}}{(\phi_{sd}^*)^{-\theta}} + \dots \end{aligned}$$

Completing the integration, rearranging terms and using equation (6), we obtain

$$\bar{G}_{sd} = f_{sd}(1)^{\tilde{\theta}} \sum_{g=1}^{\infty} Z_{sd}(g)^{-\tilde{\theta}}. \quad (13)$$

For the average number of products to be a well defined and finite number we require that

Assumption 4 Parameters are such that $\sum_{g=1}^{\infty} Z_{sd}(g)^{-\tilde{\theta}}$ is strictly positive and finite.

In Appendix C we demonstrate that fixed costs expenditure is a constant share of firm sales (where we denote means using a bar), as summarized in Proposition 3:

$$\bar{F}_{sd}/\bar{T}_{sd} = [\theta - (\sigma - 1)] / (\theta\sigma).$$

We also demonstrate that profit income and wage income can be expressed as a constant share of total output y_s :

$$\pi_s = \eta y_s, \quad w_s = (1 - \eta) y_s,$$

where $\eta \equiv 1/(\tilde{\theta}\sigma)$. Since aggregates of the model do not depend on ε , the equilibrium definition is the same as in Arkolakis and Muendler (2010a).

3 Conclusion

We have characterized the model of Arkolakis and Muendler (2010a) when the elasticity of substitution across varieties of a firm is different than the elasticity of substitution across the products of different firms. The main result is that qualitatively the extended model retains the main implications of the model where the two elasticities are the same. Future work using the structure of the generalized model to obtain estimates of the two elasticities may lead to a better understanding of the substitution effects within and across firms.

Appendix

A Optimal product prices

We characterize the first-order conditions for the firm's optimal pricing rules at every destination d . There are $G_{sd}(\phi)$ first-order conditions with respect to p_{sdg} . For any $G_{sd}(\phi)$, taking the first derivative of profits $\pi_{sd}(\phi)$ with respect to p_{sdg} and dividing by $p_{sdg}^{-\varepsilon} P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d$ yields

$$\begin{aligned} \frac{\partial \pi_{sd}(\phi)}{\partial p_{sdg}} &= P_d^{\sigma-1} T_d \cdot P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} p_{sdg}^{-\varepsilon} \left\{ 1 - \varepsilon \left(1 - \frac{w_s}{\phi/h(g)} \tau_{sd} p_{sdg}^{-1} \right) \right. \\ &\quad \left. + (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} \sum_{k=1}^{G_{sd}(\phi)} \left(p_{sdk} - \frac{w_s}{\phi/h(k)} \tau_{sd} \right) p_{sdk}^{-\varepsilon} \right\}. \end{aligned} \quad (\text{A.1})$$

The first-order conditions require that (A.1) equals zero for all products $g = 1, \dots, G_{sd}(\phi)$. Use the first-order conditions for any two products g and g' and reformulate to find

$$p_{sdg}/p_{sdg'} = h(g)/h(g').$$

So the firm must optimally charge an identical markup over the marginal costs for all products. Define this optimal markup as \bar{m} . To solve out for \bar{m} in terms of primitives, use $p_{sdg} = \bar{m} w_s \tau_{sd} / [\phi/h(g)]$ in the first-order condition above and simplify:

$$1 - \varepsilon \frac{1}{\bar{m}} + (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} \frac{\bar{m} - 1}{\bar{m}} \sum_{k=1}^{G_{sd}(\phi)} p_{sdk}^{-(\varepsilon-1)} = 0.$$

Note that $\sum_{k=1}^{G_{sd}(\phi)} p_{sdk}^{-(\varepsilon-1)} = P_{sd}(\phi; G_{sd})^{-(\varepsilon-1)}$. Solving the first-order condition for \bar{m} , we find the optimal markup over each product g 's marginal cost

$$\bar{m} = \tilde{\sigma} \equiv \sigma/(\sigma-1).$$

A firm with productivity ϕ optimally charges a price

$$p_{sdg}(\phi) = \tilde{\sigma} w_s \tau_{sd} / [\phi/h(g)] \quad (\text{A.2})$$

for its products $g = 1, \dots, G_{sd}(\phi)$.

B Second-order conditions

We now turn to the second-order conditions for price choice. To find the entries along the diagonal of the Hessian matrix, take the first derivative of condition (A.1) with respect to the own price p_{sdg} and then replace $w_s \tau_{sd} / [\phi / h(g)] = p_{sdg}(\phi) / \tilde{\sigma}$ by the first-order condition to find

$$\begin{aligned} \frac{\partial^2 \pi_{sd}(\phi)}{(\partial p_{sdg})^2} &= P_d^{\sigma-1} T_d \cdot P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} p_{sdg}^{-\varepsilon} \left\{ -\frac{\varepsilon}{\tilde{\sigma}} p_{sdg}^{-1} \right. \\ &\quad \left. + (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} [-(\varepsilon - 1) + \varepsilon / \tilde{\sigma}] p_{sdg}^{-\varepsilon} \right. \\ &\quad \left. + (\varepsilon - \sigma)(\varepsilon - 1) P_{sd}(\phi; G_{sd})^{2(\varepsilon-1)} p_{sdg}^{-\varepsilon} \cdot \sum_{k=1}^{G_{sd}} (1 - 1/\tilde{\sigma}) p_{sdk}^{-(\varepsilon-1)} \right\} \\ &= P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d \cdot \left\{ -\varepsilon p_{sdg}^{-\varepsilon-1} + (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} p_{sdg}^{-2\varepsilon} \right\} / \tilde{\sigma}. \quad (\text{B.3}) \end{aligned}$$

This term is strictly negative if and only if

$$(\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} p_{sdg}^{-(\varepsilon-1)} < \varepsilon.$$

If $\varepsilon \leq \sigma$, this last condition is satisfied because the left-hand side is weakly negative and $\varepsilon > 0$. If $\varepsilon > \sigma$, then we can rewrite the condition as $p_{sdg}^{-(\varepsilon-1)} / [\sum_{k=1}^{G_{sd}} p_{sdg}^{-(\varepsilon-1)}] < 1 < \varepsilon / (\varepsilon - \sigma)$ so that the condition is satisfied. So, the diagonal entries of the Hessian matrix are strictly negative.

To derive the entries off the diagonal of the Hessian matrix, we take the derivative of condition (A.1) for product g with respect to any other price $p_{sdg'}$ and then replace $w_s \tau_{sd} / [\phi / h(g')] = p_{sdg'}(\phi) / \tilde{\sigma}$ by the first-order condition to find

$$\begin{aligned} \frac{\partial^2 \pi_{sd}(\phi)}{\partial p_{sdg} \partial p_{sdg'}} &= P_d^{\sigma-1} T_d \cdot P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} p_{sdg}^{-\varepsilon} \left\{ (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} [-(\varepsilon - 1) + \varepsilon / \tilde{\sigma}] p_{sdg'}^{-\varepsilon} \right. \\ &\quad \left. + (\varepsilon - \sigma)(\varepsilon - 1) P_{sd}(\phi; G_{sd})^{2(\varepsilon-1)} p_{sdg'}^{-\varepsilon} \sum_{k=1}^{G_{sd}} (1 - 1/\tilde{\sigma}) p_{sdk}^{-(\varepsilon-1)} \right\} \\ &= P_d^{\sigma-1} T_d \cdot (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma+\varepsilon-1} p_{sdg}^{-\varepsilon} p_{sdg'}^{-\varepsilon} / \tilde{\sigma}. \quad (\text{B.4}) \end{aligned}$$

This term is strictly positive if and only if $\varepsilon > \sigma$.

Having derived the entries of the Hessian matrix, it remains to establish the conditions under which the Hessian is negative definite. We discern two cases. First the case of $\varepsilon \leq \sigma$ and then the case $\varepsilon > \sigma$.

B.1 Negative definiteness of Hessian if $\varepsilon \leq \sigma$

By (B.3) and (B.4), the Hessian matrix can be written as

$$\mathbf{H} = P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d [\mathbf{H}_A + (\varepsilon-\sigma)P_{sd}(\phi; G_{sd})^{\varepsilon-1} \mathbf{H}_B],$$

where

$$\mathbf{H}_A \equiv \begin{pmatrix} -\varepsilon p_{sd1}^{-\varepsilon-1} & & & \\ 0 & -\varepsilon p_{sd2}^{-\varepsilon-1} & & \\ 0 & 0 & -\varepsilon p_{sd3}^{-\varepsilon-1} & \\ \dots & & & \dots \end{pmatrix} \text{ and } \mathbf{H}_B \equiv \begin{pmatrix} p_{sd1}^{-\varepsilon} p_{sd1}^{-\varepsilon} & & & \\ p_{sd2}^{-\varepsilon} p_{sd1}^{-\varepsilon} & p_{sd2}^{-\varepsilon} p_{sd2}^{-\varepsilon} & & \\ p_{sd3}^{-\varepsilon} p_{sd1}^{-\varepsilon} & p_{sd3}^{-\varepsilon} p_{sd2}^{-\varepsilon} & p_{sd3}^{-\varepsilon} p_{sd3}^{-\varepsilon} & \\ \dots & & & \dots \end{pmatrix}.$$

The Hessian matrix \mathbf{H} is negative definite if and only if the negative Hessian

$$-\mathbf{H} = P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} P_d^{\sigma-1} T_d [-\mathbf{H}_A + (\sigma-\varepsilon)P_{sd}(\phi; G_{sd})^{\varepsilon-1} \mathbf{H}_B]$$

is positive definite. Note that the sum of one positive definite matrix and any number of positive semidefinite matrices is positive definite. So, if $-\mathbf{H}_A$ and \mathbf{H}_B are positive semidefinite and at least one of the two matrices is positive definite (given $\varepsilon \leq \sigma$), then the Hessian is negative definite.

A necessary and sufficient condition for a matrix to be positive definite is that the leading principal minors of the matrix are positive. The leading principal minors of $-\mathbf{H}_A$ are positive, so $-\mathbf{H}_A$ is positive definite. For \mathbf{H}_B , the first leading principal minor is positive, and all remaining principal minors are equal to zero. So \mathbf{H}_B is positive semidefinite. Therefore the Hessian matrix \mathbf{H} is negative definite.

B.2 Negative definiteness of Hessian if $\varepsilon > \sigma$

Another necessary and sufficient condition for the Hessian matrix \mathbf{H} to be negative definite is that the leading principal minors alternate sign, with the first principal minor being negative. The first principal minor the first diagonal entry is strictly negative as is any diagonal entry by (B.3). An application of the leading principal minor test in our case requires a recursive computation of the determinants of $G_{sd}(\phi)$ submatrices (a solution of polynomials with order up to $G_{sd}(\phi)$). We choose to check for negative definiteness of the Hessian in two separate ways when $\varepsilon > \sigma$. First, we derive a sufficient (but not necessary) condition for negative definiteness of the Hessian and query its empirical validity. Second, we present a necessary (but not sufficient) condition for negative definiteness of the Hessian for any pair of two products.

Sufficiency. A sufficient condition for the Hessian to be negative definite is due to McKenzie (1960): a symmetric diagonally dominant matrix with strictly negative diagonal entries is negative definite. A matrix is diagonally dominant if, in every row, the absolute value of the diagonal entry strictly exceeds the sum of the absolute values of all off-diagonal entries. By our derivations above, all diagonal entries of the Hessian are strictly negative.

For $\varepsilon > \sigma$, the condition for the Hessian to be diagonally dominant is

$$\sum_{\substack{k \neq g \\ k=1 \\ k=G_{sd}}} (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} p_{sdk}^{-\varepsilon} p_{sdg}^{-\varepsilon} < \varepsilon p_{sdg}^{-\varepsilon-1} - (\varepsilon - \sigma) P_{sd}(\phi; G_{sd})^{\varepsilon-1} p_{sdg}^{-2\varepsilon}$$

for all of a firm ϕ 's products (rows of its Hessian), where we cancelled the strictly positive terms $P_d^{\sigma-1} T_d P_{sd}(\phi; G_{sd})^{\varepsilon-\sigma} / \tilde{\sigma}$ from the inequality. So, for $\varepsilon = \sigma$ the Hessian is diagonally dominant.

Using the optimal price (A.2) of product g from the first-order condition and rearranging terms yields the following condition

$$\frac{\sum_{k=1}^{G_{sd}} h(k)^{-\varepsilon}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} < \frac{\varepsilon}{\varepsilon - \sigma} h(g)^{-1} \quad (\text{B.5})$$

for the Hessian to be a diagonally dominant matrix at the optimum.

By convention and without loss of generality $h(1) = 1$ for a firm with productivity ϕ . So the product efficiency schedule $h(g)$ strictly exceeds unity for the second product and subsequent products. As a result, the left-hand side of the inequality is bounded above for an exporter with a scope of at least two products at a destination:

$$\frac{\sum_{k=1}^{G_{sd}} h(k)^{-\varepsilon}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} < \frac{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} = 1.$$

So, a sufficient (but not necessary) condition for the Hessian to be negative definite is

$$1 \leq h(g) < \frac{\varepsilon}{\varepsilon - \sigma}$$

for all of the firm's products. However, the Hessian can still be negative definite even if this condition fails. Clearly, the Hessian becomes negative definite the closer is ε to σ because then the off-diagonal entries approach zero and the Hessian is trivially diagonally dominant. Moreover, the Hessian can be negative definite even if it is not a diagonally dominant matrix.

To query the empirical validity of the sufficient condition $h(g) < \varepsilon / \varepsilon - \sigma$, consider evidence on products and brands in Broda and Weinstein (2007). Their preferred estimates for ε and σ

within and across domestic U.S. brand modules are 11.5 and 7.5. Estimates in Arkolakis and Muendler (2010a) suggest that $\alpha(\varepsilon - 1)$ is around 2.56 under the specification that $h(g) = g^\alpha$. These parameters imply that the condition $h(g) < \varepsilon/\varepsilon - \sigma$ is satisfied for Hessians with up to 76 products. In the Arkolakis and Muendler (2010a) data, less than 0.37 percent of the firm-country observations involve 77 or more products in a market (with a median of one product and a mean of 3.52). Even if additional products individually violate the sufficient condition, Hessians with more products may still be negative definite.

Necessity. Consider any two products g and g' . Since negative definiteness of the Hessian must be independent of the ordering of products, these two products can be assigned the first and second row in the Hessian without loss of generality. As stated before, a necessary and sufficient condition for the Hessian to be negative definite is that the leading principal minors of the Hessian alternate sign, with the first principal minor being negative. So a necessary condition for the Hessian to be negative definite is that the principal minors of any two products (first and second in the Hessian) alternate sign, with the first principal minor negative and the second positive.

The first principle minor is strictly negative because all diagonal entries are strictly negative by (B.3). The second principal minor is strictly positive if and only if the determinant satisfies

$$2\varepsilon^2 P_{sd}(\phi; G_{sd})^{-(\varepsilon-1)} - \varepsilon(\varepsilon - \sigma) \left(p_{sdg}^{-(\varepsilon-1)} + p_{sdg'}^{-(\varepsilon-1)} \right) - (\varepsilon - \sigma)^2 (p_{sdg} p_{sdg'})^{-(\varepsilon-1)} P_{sd}(\phi; G_{sd})^{\varepsilon-1} > 0, \quad (\text{B.6})$$

where we cancelled the strictly positive terms $P_d^{2(\sigma-1)} T_d P_{sd}(\phi; G_{sd})^{2(\varepsilon-\sigma)} / \tilde{\sigma}^2$ from the inequality and multiplied both sides by $p_{sdg}^{\varepsilon-1} p_{sdg'}^{\varepsilon-1} P_{sd}(\phi; G_{sd})^{-(\varepsilon-1)}$.

To build intuition, consider the dual-product case with $G_{sd}(\phi) = 2$. Then condition (B.6) simplifies to

$$\frac{h(g)^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} \cdot \frac{h(g')^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} < \frac{\varepsilon}{\varepsilon - \sigma} \frac{\varepsilon + \sigma}{\varepsilon - \sigma}.$$

For $\varepsilon > \sigma$, both terms in the product on the right-hand side strictly exceed unity while the terms in the product on the left-hand side are strictly less than one, and the condition is satisfied.

In the multi-product case with $G_{sd}(\phi) > 2$, replace $p_{sdg}^{-(\varepsilon-1)} + p_{sdg'}^{-(\varepsilon-1)} = P_{sd}(\phi; G_{sd})^{-(\varepsilon-1)} - \sum_{k \neq g, g'} p_{sdk}^{-(\varepsilon-1)}$ in condition (B.6) and simplify to find

$$\frac{h(g)^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} \cdot \frac{h(g')^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}} < \frac{\varepsilon}{\varepsilon - \sigma} \frac{\varepsilon + \sigma}{\varepsilon - \sigma} + \frac{\varepsilon}{\varepsilon - \sigma} \frac{\sum_{k \neq g, g'} h(k)^{-(\varepsilon-1)}}{\sum_{k=1}^{G_{sd}} h(k)^{-(\varepsilon-1)}}.$$

For $\varepsilon > \sigma$, the necessary condition on any two products of a multi-product firm is trivially satisfied by the above derivations because the additional additive term on the right-hand side is strictly positive.

In summary, parameters of our model are such that, for any two products of a multi-product firm, the second-order condition is satisfied.

C Proof of Proposition 3

Average sales from s to d are

$$\bar{T}_{sd} = \int_{\phi_{sd}^*} y_{sd}(G_{sd}) \frac{\theta (\phi_{sd}^*)^\theta}{\phi^{\theta+1}} d\phi = \sigma f_{sd}(1) \theta \int_{\phi_{sd}^*} \frac{\phi^{\sigma-2-\theta} / (\phi_{sd}^*)^{\sigma-1-\theta}}{H(G_{sd}(\phi))^{\sigma-1}} d\phi.$$

The proof of the proposition follows from the following Lemma.

Lemma 1 *Suppose Assumptions 1, 2 and 3 hold. Then*

$$\int_{\phi_{sd}^*} \frac{\phi^{\sigma-2-\theta} / (\phi_{sd}^*)^{\sigma-1-\theta}}{H(G_{sd}(\phi))^{\sigma-1}} d\phi = \frac{f_{sd}(1)^{\tilde{\theta}-1}}{\theta - (\sigma-1)} \tilde{F}_{sd},$$

where

$$\tilde{F}_{sd} \equiv \sum_{v=1}^{\infty} \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{[H(v)^{1-\sigma} - H(v-1)^{1-\sigma}]^{-\tilde{\theta}}}.$$

Proof. Note that

$$\begin{aligned} \int_{\phi_{sd}^*} \frac{\phi^{\sigma-2-\theta} / (\phi_{sd}^*)^{\sigma-1-\theta}}{H(G_{sd}(\phi))^{\sigma-1}} d\phi &= H(1)^{1-\sigma} \int_{\phi_{sd}^*}^{\phi_{sd}^{*,2}} \phi^{\sigma-2-\theta} d\phi + H(2)^{1-\sigma} \int_{\phi_{sd}^{*,2}}^{\phi_{sd}^{*,3}} \phi^{\sigma-2-\theta} d\phi + \dots \\ &= H(1)^{1-\sigma} \left[\frac{(\phi_{sd}^{*,2})^{\sigma-1-\theta} - (\phi_{sd}^*)^{\sigma-1-\theta}}{[\theta - (\sigma-1)] (\phi_{sd}^*)^{\sigma-1-\theta}} \right] \\ &\quad + H(2)^{1-\sigma} \left[\frac{(\phi_{sd}^{*,3})^{\sigma-1-\theta} - (\phi_{sd}^{*,2})^{\sigma-1-\theta}}{[\theta - (\sigma-1)] (\phi_{sd}^*)^{\sigma-1-\theta}} \right] + \dots \end{aligned}$$

Also note that, using equations (4) and (6), the ratio $\left[(\phi_{sd}^{*,2})^{\sigma-1-\theta} - (\phi_{sd}^*)^{\sigma-1-\theta} \right] / (\phi_{sd}^*)^{\sigma-1-\theta}$ can be rewritten as

$$\begin{aligned} & \frac{(\phi_{sd}^{*,G})^{\sigma-1-\theta} - (\phi_{sd}^{*,G-1})^{\sigma-1-\theta}}{(\phi_{sd}^*)^{\sigma-1-\theta}} = \\ &= \frac{\left[\frac{(\phi_{sd}^*)^{\sigma-1}}{H(g)^{1-\sigma} - H(g-1)^{1-\sigma}} \frac{f_{sd}(g)}{f_{sd}(1)} \right]^{\frac{\sigma-1-\theta}{\sigma-1}} - \left[\frac{(\phi_{sd}^*)^{\sigma-1}}{H(g-1)^{1-\sigma} - H(g-2)^{1-\sigma}} \frac{f_{sd}(g-1)}{f_{sd}(1)} \right]^{\frac{\sigma-1-\theta}{\sigma-1}}}{\left[(\phi_{sd}^*)^{\sigma-1} \right]^{\frac{\sigma-1-\theta}{\sigma-1}}} \\ &= f_{sd}(1)^{\tilde{\theta}-1} \left\{ \frac{f_{sd}(g)^{1-\tilde{\theta}}}{[H(g)^{1-\sigma} - H(g-1)^{1-\sigma}]^{1-\tilde{\theta}}} - \frac{f_{sd}(g-1)^{1-\tilde{\theta}}}{[H(g-1)^{1-\sigma} - H(g-2)^{1-\sigma}]^{1-\tilde{\theta}}} \right\}. \end{aligned}$$

We define²

$$\begin{aligned} \tilde{F}_{sd} &\equiv \sum_{v=1} H(v)^{1-\sigma} \left[\frac{[f_{sd}(v+1)]^{1-\tilde{\theta}}}{[H(v+1)^{1-\sigma} - H(v)^{1-\sigma}]^{1-\tilde{\theta}}} - \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{[H(v)^{1-\sigma} - H(v-1)^{1-\sigma}]^{1-\tilde{\theta}}} \right] \\ &= \sum_{v=1} \left[[H(v)^{1-\sigma} - H(v-1)^{1-\sigma}] \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{[H(v)^{1-\sigma} - H(v-1)^{1-\sigma}]^{1-\tilde{\theta}}} \right] \\ &= \sum_{v=1} \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{[H(v)^{1-\sigma} - H(v-1)^{1-\sigma}]^{-\tilde{\theta}}}. \end{aligned}$$

With this definition we obtain

$$\int_{\phi_{sd}^*} \frac{\phi^{\sigma-2-\theta} / (\phi_{sd}^*)^{\sigma-1-\theta}}{H(G_{sd}(\phi))^{\sigma-1}} d\phi = \frac{f_{sd}(1)^{\tilde{\theta}-1}}{\theta - (\sigma-1)} \tilde{F}_{sd}.$$

²In the special case with $\varepsilon = \sigma$, we can rearrange the terms and find

$$\tilde{F}_{sd} = \sum_{v=1}^{\infty} \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{[h(v)^{\sigma-1}]^{-\tilde{\theta}}} = \sum_{v=1}^{\infty} \frac{[f_{sd}(v)]^{1-\tilde{\theta}}}{h(v)^{-\theta}}.$$

D Welfare

We have that

$$\begin{aligned}
P_d^{1-\sigma} &= \sum_s \int_{\phi_{sd}^*} [P_{sd}(\phi)]^{1-\sigma} \mu(\phi) \mathbf{d}\phi \\
&= \sum_s \int_{\phi_{sd}^*} M_{sd} \left[\sum_{v=1}^{G_{sd}(\phi)} \left(\tilde{\sigma} \frac{w_s}{\phi/h(g)} \tau_{sd} \right)^{1-\varepsilon} \right]^{\frac{1-\sigma}{1-\varepsilon}} \frac{\theta (\phi_{sd}^*)^\theta}{\phi^{\theta+1}} \mathbf{d}\phi \\
&= \sum_s (\tilde{\sigma} w_s \tau_{sd})^{1-\sigma} b_s^\theta \theta \left[H(1)^{1-\sigma} \left(\frac{(\phi_{sd}^{*,2})^{\sigma-1-\theta} - (\phi_{sd}^{*,1})^{\sigma-1-\theta}}{\theta - (\sigma-1)} \right) + \dots \right] \\
&= \sum_s (\tilde{\sigma} w_s \tau_{sd})^{-\theta} b_s^\theta \theta \left(\frac{f_{sd}(1)}{\frac{1}{\sigma} T_d} \right)^{1-\tilde{\theta}} \left[H(1)^{1-\sigma} \left(\frac{(\phi_{sd}^{*,2})^{\sigma-1-\theta} - (\phi_{sd}^{*,1})^{\sigma-1-\theta}}{(\phi_{sd}^{*,1})^{\sigma-1-\theta}} \right) + \dots \right],
\end{aligned}$$

where we use the definition of $\phi_{sd}^{*,1}$ for the last step. The final term in parentheses equals \tilde{F}_{sd} so

$$P_d^{-\theta} = \frac{\theta (\tilde{\sigma})^{-\theta}}{\left(\frac{1}{\sigma}\right)^{1-\theta/(\sigma-1)} T_d^{1-\tilde{\theta}}} \sum_s b_s^\theta (w_s \tau_{sd})^{-\theta} \tilde{F}_{sd}.$$

Using this relationship in equation (12), we obtain

$$\left(\frac{T_d}{P_d} \right)^\theta = \left(\frac{T_d}{w_d} \right)^\theta \frac{\theta (\tilde{\sigma})^{-\theta}}{(\sigma)^{\tilde{\theta}-1}} \frac{b_d^\theta}{\lambda_{dd}^{-\theta}} \frac{\tilde{F}_{dd}(1)}{T_d^{1-\tilde{\theta}}}.$$

If trade is balanced then $T_d = y_d$, where y_d is output per capita. Since, by the definition of $\tilde{F}_{dd}(1)$, this variable is homogeneous of degree $1 - \tilde{\theta}$ in wages and w_d/y_d is constant in all equilibria (see proof below) we arrive at the same welfare expression as in Arkolakis, Costinot, and Rodríguez-Clare (2010): the share of domestic sales in expenditure λ_{dd} and the coefficient of the Pareto distribution are sufficient statistics to characterize aggregate welfare in the case of balanced trade.

The final step is to verify that the wage w_d is a constant fraction of per-capita output y_d so that the first ratio on the right-hand side is constant. We demonstrate this below.

E Proof that wage is constant fraction of output per capita

We show that the ratio w_d/y_d is a constant number. We first look at the share of fixed costs in bilateral sales. Average fixed costs incurred by firms from s selling to d are

$$\begin{aligned}\bar{F}_{sd} &= \int_{\phi_{sd}^*}^{\phi_{sd}^{*,2}} F_{sd}(1) \theta \frac{(\phi_{sd}^*)^\theta}{\phi^{\theta+1}} d\phi + \int_{\phi_{sd}^{*,2}}^{\phi_{sd}^{*,3}} F_{sd}(2) \theta \frac{(\phi_{sd}^*)^\theta}{\phi^\theta} d\phi + \\ &= -F_{sd}(1) (\phi_{sd}^*)^\theta \left[(\phi_{sd}^{*,2})^{-\theta} - (\phi_{sd}^*)^{-\theta} \right] - F_{sd}(2) (\phi_{sd}^*)^\theta \left[(\phi_{sd}^{*,3})^{-\theta} - (\phi_{sd}^{*,2})^{-\theta} \right] - \dots\end{aligned}$$

Using the definition $F_{sd}(G_{sd}) = \sum_{g=1}^{G_{sd}} f_{sd}(g)$ and collecting terms with respect to $\phi_{sd}^{*,G}$ we can write the above expression as

$$\bar{F}_{sd} = f_{sd}(1) + (\phi_{sd}^{*,2})^{-\theta} (\phi_{sd}^*)^\theta f_{sd}(2) + (\phi_{sd}^{*,3})^{-\theta} (\phi_{sd}^*)^\theta f_{sd}(3) + \dots$$

Using the definition of $\phi_{sd}^{*,G}$ from equation (6) to replace terms in the above equation, we obtain

$$\left(\phi_{sd}^{*,G} \right)^{\sigma-1} = \frac{(\phi_{sd}^*)^{\sigma-1}}{H(G_{sd})^{-(\sigma-1)} - H(G_{sd}-1)^{-(\sigma-1)}} \frac{f_{sd}(G_{sd})}{f_{sd}(1)}.$$

Thus,

$$\begin{aligned}\bar{F}_{sd} &= f_{sd}(1) + \left(\frac{f_{sd}(2)^{1/(\sigma-1)} \left[H(2)^{-(\sigma-1)} - H(1)^{-(\sigma-1)} \right]^{-1/(\sigma-1)}}{f_{sd}(1)^{1/(\sigma-1)} \left[H(1)^{-(\sigma-1)} \right]^{-1/(\sigma-1)}} \right)^{-\theta} f_{sd}(2) + \dots \\ &= \left[f_{sd}(1) + f_{sd}(1)^{\tilde{\theta}} \left(f_{sd}(2)^{1/(\sigma-1)} \left[H(2)^{-(\sigma-1)} - H(1)^{-(\sigma-1)} \right]^{-1/(\sigma-1)} \right)^{-\theta} f_{sd}(2) + \dots \right] \\ &= [f_{sd}(1)]^{\tilde{\theta}} \left[f_{sd}(1)^{1-\tilde{\theta}} + \frac{f_{sd}(2)^{1-\tilde{\theta}}}{\left[H(2)^{-(\sigma-1)} - H(1)^{-(\sigma-1)} \right]^{-\tilde{\theta}}} + \frac{f_{sd}(3)^{1-\tilde{\theta}}}{\left[H(3)^{-(\sigma-1)} - H(2)^{-(\sigma-1)} \right]^{-\tilde{\theta}}} \dots \right] \\ &= [f_{sd}(1)]^{\tilde{\theta}} \tilde{F}_{sd}\end{aligned}$$

and therefore

$$\frac{\bar{F}_{sd}}{\bar{T}_{sd}} = \frac{f_{sd}(1)^{\tilde{\theta}} \left[f_{sd}(1)^{1-\tilde{\theta}} + \frac{f_{sd}(2)^{1-\tilde{\theta}}}{h(2)^\theta} + \dots \right]}{\frac{f_{sd}(1)^{\tilde{\theta}} \theta \sigma}{\theta - (\sigma-1)} \sum_{g=1}^{\infty} \frac{f_{sd}(g)^{1-\tilde{\theta}}}{h(g)^\theta}} = \frac{\theta - (\sigma-1)}{\theta \sigma} \frac{\sum_{g=1}^{\infty} \frac{f_{sd}(g)^{1-\tilde{\theta}}}{h(g)^\theta}}{\sum_{g=1}^{\infty} \frac{f_{sd}(g)^{1-\tilde{\theta}}}{h(g)^\theta}} < 1.$$

Finally, the share of profits generated by the corresponding bilateral sales is the share of variable profits in total sales ($1/\sigma$) minus the average fixed costs paid, as derived above. So

$$\frac{\bar{\pi}_{sd}}{\bar{T}_{sd}} = \frac{1}{\sigma} - \frac{\theta - (\sigma-1)}{\theta \sigma} = \frac{\sigma-1}{\theta \sigma} = \frac{1}{\tilde{\theta} \sigma} \equiv \eta.$$

This finding implies that the wage is a constant fraction of per capita income. To see why observe that total profits for country s are $\pi_s L_s = \sum_k \lambda_{sk} T_k / (\tilde{\theta}\sigma)$, where $\sum_k \lambda_{sk} T_k$ is the country's total income because total manufacturing sales of a country s equal its total sales across all destinations. So profit income and wage income can be expressed as constant shares of total income:

$$\pi_s L_s = \frac{1}{\tilde{\theta}\sigma} Y_s \quad \text{and} \quad w_s L_s = \frac{\tilde{\theta}\sigma - 1}{\tilde{\theta}\sigma} Y_s.$$

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