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**THE REAL FOUNDATIONS OF THE ALLEGED ERRORS IN
ALLAIS' IMPOSSIBILITY THEOREM :
UNCEASINGLY REPEATED ERRORS OR CONTRADICTIONS
OF MARK MACHINA**

ABSTRACT

- 1.- *The Impossibility Theorem rests on five fundamental properties. On each of them Machina has confirmed his agreement over and over again between 1984 and 1986. The proof of the Impossibility Theorem is mathematically and rigorously correct.*

- 2.- *The alleged Machina's demonstration of the Allais' two errors is indeed only grounded on a series of errors and it is in total contradiction with the interpretations explicitly given by Machina from 1982 to 1986 of his local utility theory in the discrete case, especially in his letters.*

- 3.- In fact one cannot but ask if Machina really understands the meaning and the implications of his formulation of the local utility, and what is really in question.

In any case it would be without any doubt eminently desirable that before hastily concluding that the others are making mistakes, Machina should begin by seriously asking himself if he is not himself into error.

Mark Machina's Paper

1.- The Editor of "*Theory and Decision*", Bertrand Munier, sent me on May 27, 1994, a paper by Mark Machina with an attractive and provocative title : "*Two errors in the Allais Impossibility Theorem*". It would be impossible, indeed, not to answer such a paper for as the saying goes, "*Qui ne dit mot consent*" (silence gives consent).

It is indeed only amazing that it took *more than ten years* to Machina to answer the criticisms which I kept addressing to him from 1983 until 1986. *If my alleged errors were so obvious, why did it take so many years before they could be shown ?*

1.1 As a matter of fact, I haven't been able until now to answer Machina's paper. My time has been entirely used up, on one hand by the task of editing the first printed version of my 1943 work, in view of which I have awarded the 1988 Nobel Prize in Economic Science, with a new and long introduction ¹, and on another hand by the task of publishing an important book on Europe ². Quite obviously, facing the choice to respond to a paper resting on erroneous allegations or to publish works having, in my view, an exceptional significance, I could not hesitate.

Still today, I am completely wrapped up in a new edition of my book "*Economie et Intérêt*", which was effective also in the Nobel Award ³, in a new edition of my "*Autoportraits*" ⁴ with substantial additions, and finally in writing an important whole survey of my works in physics and on anisotropy of space ⁵.

1.2 It appears *however necessary and possible* to provide the reader with some major observations on what is effectively in question by referring to my 1983 memoir "*The Foundations of the Theory of Utility and Risk*", and to my 1986 memoir, "*A New Neo-Bernoullian Theory : The Machina Theory. A Critical Analysis*", as well as to the *Postscript* which came with it.

In fact, the alleged refutation by Machina rests fundamentally on mere errors and it is in total contradiction with the interpretations of his theory of local utility as he himself formulated them from 1982 on until 1986, especially in his correspondence.

Again as from 1983 to 1986, Machina's paper, due to its wording, makes obscure *what really is in question* and uselessly complicates *perfectly clear questions*. I shall give hereunder some illustrative examples of it. Pointing out to the following observations will enable the reader to easily perceive the errors made by Machina *with respect to the interpretations of his theory such as he formulated them in his writings and his correspondence prior the writing of my 1986 memoir and 1987 Postscript.*

In any case, it is *absolutely necessary* that the reader first reads *very carefully and completely* my 1986 memoir (including the notes) and the *Postscript* to it ⁶. That memoir, which was *entirely written* in 1986, (a 28 pages first draft of which was circulated during the 1986 "*Third Conference on the Foundations and Applications of Utility. Risk and Decision Theories*", in Aix-en-Provence), was only published in 1988.

The final version of that memoir was entirely completed on November 3, 1986, and the *Postscript*, written on the suggestion of Bertrand Munier, was completed on April 27, 1987. These papers could only take into account the observations which Machina presented *before August 27, 1986*, date by which I received the last letter which Machina wrote to me ⁷.

1.3 The reader will understand that I cannot accept to spend too much of a scarce time to correct Machina's mistakes, *line after line*, as I did not stop doing during three years, from 1983 to 1986, whatever their present presentation.

What matters is only that the reader be enabled to make his own judgment after a thorough reading of my 1986 memoir and of the *Postscript* to it, and in view of my observations on the assertions expressed today by Machina. *Paralogisms, sophisms, mistakes and contradictions of Machina will by then without any doubt appear very clearly to him.*

The effective foundations of the proof of the Impossibility Theorem. Five fundamental properties

2.- The proof of the Impossibility Theorem *rests on five fundamental properties* of the local utility theory in the discrete case, *all formally recognized by Machina* as precised in the following § 3 ⁸.

2.a Domain of definition of the local utility index in the discrete case

"According to Machina the local neo-Bernoullian index $\pi(x; p)$ in the discrete case is well defined over the whole range

$$(1) \qquad 0 \leq x \leq M \qquad (11)$$

in which M represents the maximum value of x (1982, p. 294)". (M. 350 and 352)

2.b Definition of the local utility in the discrete case

"The most general preference function is written in the discrete case

$$(2) \quad \phi = H(x_1, \dots, x_n, p_1, \dots, p_n) \quad (20^*)$$

H is a function of the x_i, p_i, n being any integer". (M. 355)

• *"In the discrete case Machina's "local utility" theory boils down by defining the local utility function by the definition equation*

$$(3) \quad d_p \phi = \sum_i \pi(x_i; p) dp_i \quad (23^*)$$

in which $\pi(x_i; p)$ is the value of the function $\pi(x; p)$ for $x = x_i$." (M. 355)

The definition relation (3) means that every function $\pi(x; p)$ which satisfies that relation can be considered as a *determination of the local utility in Machina's sense*, in the discrete case, and, reciprocally, every determination of local utility must satisfy this relation.

In fact, local utility is defined by relation (3) for *given* values of n and of the (x_i, p_i) at the (x_i, p_i) point. $d_p \phi$ is the first order differential of the variation $\delta \phi$ of ϕ when, around the (x_i, p_i) point, the p_i vary by δp_i , the first order differentials of which are the dp_i . Naturally we have $\sum_i \delta p_i = 0$. Thus the variations δp_i are not independent.

When one considers *without any further specification* a subject whose random psychology, *when facing a lottery* for example, is represented by the discrete relation (2), *one assumes by the same token indeed that one has no other information whatsoever on his choice field.*

2.c Definition of the local neo-Bernoullian index up to a linear transformation in the discrete case

"In the more general continuous case, *according to Machina*, if two calculations lead to two different expressions $\pi(x; \varphi)$ and $\pi^*(x; \varphi)$ of the local utility index we necessarily have

$$(4) \quad \pi^*(x; \varphi) = \lambda(\varphi) \pi(x; \varphi) + \mu(\varphi) \quad (12)$$

in which $\lambda(\varphi)$ and $\mu(\varphi)$ are functionals of φ independent of x ". (M. 350)

"In the discrete case this relation writes

$$(5) \quad \pi^*(x; p) = \lambda \pi(x; p) + \mu \quad (\text{A } 23)$$

in which λ and μ are functions of the x_i, p_i , independent of x ". (M. 381)

2.d Expression of the preference function ϕ as a function of its moments

"A discrete distribution of order n is fully determined if its $2n - 1$ first moments are given, and conversely. Consequently, the preference function (20*) can always be expressed in the general form

$$(6) \quad \phi = G(M_1, \dots, M_k, \dots, M_{2n-1}) \quad (31)$$

with

$$(7) \quad M_k = \sum_i p_i x_i^k \quad (32) \quad \text{" (M.357)}$$

2.e Determination of the local utility function from the preference function as a function of its moments

"From relations (31) and (32) we deduce

$$(8) \quad d_p \phi = \sum_i \left[\sum_k \frac{\partial G}{\partial M_k} \frac{\partial M_k}{\partial p_i} \right] dp_i = \sum_i \left[\sum_k \frac{\partial G}{\partial M_k} x_i^k \right] dp_i \quad (40)$$

in which the $d_p \phi$ represents the differential of ϕ with respect to the p_i alone.

"As a result it follows from the definition relation (23*) that the corresponding expression of the local neo-Bernoullian index is according to Machina's theory.

$$(9) \quad \pi(x; p) = \sum_k \frac{\partial G}{\partial M_k} x^k \quad (42) \quad (\text{or A } 20) \quad \text{" (M. 359)}$$

"Since from (31) we have in the discrete case

$$(10) \quad d_p \phi = \sum_i \frac{\partial G}{\partial p_i} dp_i \quad (45)$$

we have from (23*)

$$(11) \quad \pi(x_i; p) = \partial G / \partial p_i \quad (46)$$

"Thus, the expression (42) of the local neo-Bernoullian index is deduced, up to within a constant, by replacing the variable x_j by the variable x in any derivative $\partial G / \partial p_j$. By reason of the symmetrical expression of the moments M_k as functions of the x_i , p_j , we obtain an expression for the index $\pi(x; p)$ which is independent of the variable x_j considered⁹.

"When the preference function ϕ is not expressed as a function of the moments, there is no direct way of calculating the neo-Bernoullian index from the preference function (20*) and the only way of doing this calculation is to start by expressing the preference function as a function of moments in the form (31)¹⁰. This can always be done, and consequently, the local index in Machina's sense can always be determined" (M. 360)

n.9. Allais, 1983, n. 102, p.98
n. 10. id., n. 102*, p. 98" (M. 393).

Indeed the reference to notes (102) and (102*) of my 1983 memoir is here altogether essential.

In fact the notations $\pi(x_j; p)$ and $\pi(x; p)$ are symbolic notations and we have

$$(11') \quad \pi(x_j; p) = \pi(x_j; M_1, M_2, \dots, M_{2n-1})$$

$$(11'') \quad \pi(x; p) = \pi(x; M_1, M_2, \dots, M_{2n-1})$$

Confirmation by Machina of the validity of the five fundamental properties (a) to (e)

3.- *The full validity* of the five fundamental properties (a) to (e) above results of Machina's correspondence prior to November 3, 1986, date at which the final version of my 1986 memoir was written⁹.

3.0 Two preliminary observations

Before I can proceed with the analysis of these five properties in view of Machina's correspondence, it seems necessary to underline two points :

1- First, as in his new memoir Machina does not say nowhere that he changed his view with respect to his prior interpretation of his theory in the discrete case, it must be considered indeed that the statements made by him in his correspondence, prior November 3, 1986, retain their entire validity.

2- Second, and as I stated it in my 1986 Memoir ¹⁰ :

"While Machina presented different examples of calculation of the "local utility" in the continuous case (1982, p. 295, and 1983 a, p. 274-275), by contrast, he has *never* given an illustration of the calculation of the "local utility" in the discrete case.

To shed light on Machina's theory and to bridge this gap, I presented in Allais (1983) the above calculation of "local utility" *in Machina's sense* in the discrete case, *when the preference function is expressed as a function of the moments* of the distribution considered ¹².

At various times in 1984 and 1986, Machina *confirmed* that expressions (42) and (46) of "local utility", deduced from expression (31) of the preference function as a function of moments and from the relation of definition (23*), *does correspond to his theory considered in the discrete case* ¹³. (M. 360-361)

n.12. Allais, 1983, p. 47, § a.6, and n. 102, 103 and 113*, p. 98-101.

[on p. 47, relation 47, read : - l u (x) .., instead of : + l u (x) ..]

n.13. His letters of January 19, 1984, January 17 and February 17, 1986". (M. 393)

Having made these two essential, and *really fundamental* observations, I proceed now with the analysis of the confirmations *expressed by Machina* regarding the five properties (a) to (e). I think I must *forcefully stress again for the sake of the reader that Machina never gave any illustration of the calculation of local utility in the discrete case.*

3.a Property (a)

Property (a) was formally stated in Machina' 1982 paper, p. 294, *as valid in all cases.*

3.b Property (b)

On January 9, 1984, I sent a letter (4 pages) to Machina with two Annexes, together with the first version of my 1983 memoir, "The Foundations of the Theory of Utility and Risk", and asked him for his observations.

In these two *Annexes*, written in November 1983 and respectively entitled "*On the Machina Formulation*" and "*The Machina Formulation in the more general discrete Case*", I made use of the *definition relation (3) above (§ 2.b)* of local utility as well as of *expression (6) above (§ 2.d)* of the preference function and of the *determination (9) (§ 2.e)* above of local utility.

Annex I gave an illustration of the calculation of local utility which I used again in § 2.3.1.a.5 of my 1983 memoir (p. 46).

Annex II precised in the most general discrete case the calculation of the local utility from the expression of the preference function *in terms of its moments*. That calculation was used again in § 2.3.1.a.6 (p. 47) of my 1983 memoir ¹¹.

- *The only observation which Machina sent to me about these two annexes in his January 19, 1984, letter is reproduced hereunder and it does only pertain to Annex I. In view of property (c) below, Machina's observation is in fact absolutely essential and, to tell the truth, quite fundamental, as indicated in note (5) of my 1986 memoir :*

"This is a basic issue to which I drew Machina's attention in my letter of January 9, 1984. In his answer of January 19, 1984, he wrote :

"Here you refer to "the indeterminacy of the local utility function". I agree with you that the local utility function is only determined up to multiplication by a positive constant or addition of a constant".

"Curiously, enough Machina did not mention in any way my letter of January 9, 1984, in his (1986) memoir". (M. 392)

In his 1994 paper, Machina again *does not make any reference to it*.

In fact, as well in the two *Annexes I and II* to my January 9, 1984, letter, *Annexes against which Machina had raised no objection*, as in my 1983 memoir, "*The Foundations of the Theory of Utility and Risk*" (§ 2.3.1., p. 42-55, and notes 92-120*, p. 97-105), I made use of properties (a), (b), (d) and (e), mentioned above (§ 2).

Following the communication to Machina on January 11, 1984, of my 1983 memoir, *no observation* was made by him on properties (b), (d) and (e) used at several points bearing on local utility of my 1983 memoir.

- *Having raised no objection in 1984, other than the one mentioned above, on the two Annexes I and II of my letter of January 9, 1984 and on my 1983 memoir, Machina has thus approved the propositions (b), (d), and (e). This agreement has been confirmed on several occasions later on ¹².*

3.c Property (c)

Machina himself, in his letter of January 19, 1984, and when answering my example of its Annex I, precised that local utility is determined only up to a multiplicative factor and up to a constant, i.e. is defined only up to a linear transformation, as I just recalled above in § 3.b. That example was identical in the discrete case to the illustration I provided of the indeterminacy of local utility in my 1986 memoir, published in 1988, § 2.2.1. (p. 356).

As I had stressed in my 1986 memoir (§ 3.b above), the January 19, 1984, statement by Machina has a *very exceptional importance in view of his later assertions*. It means indeed that, in the discrete case, local utility is determined only up to a linear transformation for any value of x , as is made clear in relation (5) of § 2.c above¹³, and as it results from the property 2 a of § 2.a above.

3.d Property (d)

Following my letter of March 18, 1986, I sent to Machina my 1986a memoir "*On the Determination of a Discrete Distribution of Order n from $2n-1$ Successive Moments*". Upon that communication, I received no observation whatsoever from Machina. *He thus confirmed his agreement with property (d).*

3.e Property (e)

1- As indicated above (§ 3.b), Machina expressed indeed *his full agreement with property (e)* as early as his letter of January 19, 1984. That agreement has been confirmed in a *particularly explicit way* in Machina's letter of February 17, 1986 (p. 2-3) concerning the § 2.3.a.1.6 (p. 47) of my 1983 memoir, "*The Foundations of Risk and Utility Theory*", as follows :

"Your second question concerns my agreement with your formulation of the local utility function in the case of discrete distributions, i.e. equation 47 on page 47 of Hagen and Wenstop, *Progress in Utility and Risk Theory*.

Although I agree with the equation 45 of the previous page, it seems as if you have made a slight error in the derivation of equation 47 (the one you asked me about). Using your notation, it seems as if this equation should read :

$$\pi(x; R) = u(x) + \sum_1 \frac{\partial f}{\partial \mu_1} \left\{ [u(x) - \bar{u}]^1 - 1 u(x) \sum_k p_k (u_k - \bar{u})^{1-1} \right\} \quad (47)$$

In other words, the negative sign indicated by my arrows was accidentally written as a positive sign in your paper. The reason that it is negative here is that \bar{u} enters negatively into the formula for u_1 . Needless to say, please let me know if you disagree with me on this point! If I am correct, I apologize for not noticing this the first time I read your paper". ¹⁴

In my letter of March 18, 1986 (p. 1-2) I thanked Mark Machina as follows :

"Thank you for the correction of the error of sign in relation 47, p. 47, of my 1984 memoir. This same error is found again in relation (2) of note 102, p. 98, but the sign is effectively correct in relation (2), note 103, p. 98. The two above errors result from an unfortunate misprint in a preceding manuscript.

"What is essential is the renewal of your general agreement of your letter of September 10, 1984, on my calculation of the local utility in the discrete case. I wanted to be sure before sending to you my memoir 1986b, "Impossibility of the Validity of Machina's Formulation of Local Utility in Any Finite Domain".

In fact, the calculation of the expression (47) above of my 1983 memoir was justified by the note (102) (p. 98) as follows :

n. 102. The function $\pi(x; R)$ of relation (47) is deduced by replacing in each derivative $\partial \phi / \partial p_j$ the g_j corresponding to p_j by x ".

With the notations of the present memoir this note means that the local utility $\pi(x; p)$ is obtained by replacing the x_j by x in each expression $\pi(x_j; p)$.

Therefore indeed the agreement given by Machina on relation (47) above confirms entirely at the same time the properties corresponding to the relations (3), (6), and (11) of § 2.b, 2.d, and 2.e above.

2- *The agreement thus given by Machina in his letter of February 17, 1986, is indeed all the more important, as in itself it confirms at the same time properties (b), (d), and (e) of § 2 above. In view of § 8.5.5 hereunder, one should stress that expression (47) above is indeed a polynomial, as any expression of local utility deduced from the consideration of moments.*

I had stressed *that very importance* in the *Postscript* (p. 399-400) of my 1986 memoir, published in 1988, on Machina's theory :

"a.- The proof of the Impossibility Theorem is based first on the calculation of local utility in the discrete case which I presented in Allais (1983), and which Machina has confirmed over and over again ...

b.- Machina's full agreement, confirmed several times since 1984, with this method of calculation is *the more significant* as it was he who brought to my attention the misprint, rectified in note 12 above, in Allais (1983)".

3- It is *absolutely essential* to underline here that in relation (47) indicated above the value x_i of x of the definition relation (3) above (§ 2.b) of local utility is *indeed replaced by the variable x* .

Therefore it results that *for Machina* any function $\pi(x; p)$ satisfying the definition relation (3) (§ 2.b) of local utility for $x = x_i$ represents *a determination of local utility for the values x different of x_i* .

Therefore, considering the indications of his letter of February 17, 1986, it results that *according to Machina* the expression (9) above (§ 2.e) (relation 42 of my 1986 memoir) *represents effectively the local utility in Machina's sense over the whole interval $(0 \leq x \leq M)$, and it is effectively a polynomial*.

Implications of the five fundamental properties. The Impossibility Theorem

4.- It is easy to deduce what the implications of the five fundamental properties are. The *Impossibility Theorem* finally results from these implications. This proof is *purely mathematical* and it can be *immediately and rigorously* deduced from the five fundamental properties (a) to (e) of Section 2 above, *confirmed again and again by Machina* (§ 3 above).

4.1 General relation between two different expressions π and π^* in the discrete case

"If, using two different calculation methods, two different expressions $\pi(x; p)$ and $\pi^*(x; p)$ of the local index are obtained, we should have from (23*)

$$(12) \quad \sum_i [\pi^*(x_i, p_i) - \pi(x_i, p_i)] dp_i = 0 \tag{33}$$

whatever the dp_i under the sole condition

$$(13) \quad \sum_i dp_i = 0 \tag{34}$$

"Consequently, we necessarily have the n relations

$$(14) \quad \pi^*(x_i, p) - \pi(x_i, p) = K [M_1 \dots, M_k \dots, M_{2n-1}] \tag{35}$$

in which K is a function of the moments which characterize the local situation considered.

"This is a really essential theorem that Machina has not taken into account 7. In particular, it flows from this theorem that one cannot infer from the n relations (35), which are only valid for the n values considered x_j of x, that

$$(15) \quad \pi^*(x; p) - \pi(x; p) = K [M_1 \dots, M_k \dots, M_{2n-1}] \quad (36)$$

whatever the value of x" (M. 357-358).

"n.7. In fact, in his letter of April 10, 1986, Machina states that $K = 0$. This is a fundamental error which does not take account of the fact that the p_i are not independent variables (see Allais, 1986d and 1986c)" (M. p. 392).

On this point see also § 5 hereunder.

4.2 Another expression $\pi^*(x; p)$ of local utility

"Now, considering the transformation

$$(16) \quad y_i = f(x_i) \quad (0 \leq f(x_i) \leq f(M)) \quad (37)$$

in which f is any increasing function, the distribution of the y_j is perfectly defined if its 2n-1 first moments

$$(17) \quad \mathcal{M}_k = \sum_i p_i f^k(x_i) \quad (1 \leq k \leq 2n-1) \quad (38)$$

are given so that the preference function ϕ defined by (20*) can also be expressed in the form

$$(18) \quad \phi = G^*(\mathcal{M}_1, \dots, \mathcal{M}_k, \dots, \mathcal{M}_{2n-1}) \quad (39)$$

"From relations (31), (32), (38), and (39), we deduce

$$(19) \quad d_p \phi = \sum_i \left[\sum_k \frac{\partial G^*}{\partial \mathcal{M}_k} \frac{\partial \mathcal{M}_k}{\partial p_i} \right] dp_i = \sum_i \left[\sum_k \frac{\partial G^*}{\partial \mathcal{M}_k} f^k(x_i) \right] dp_i \quad (41)$$

"As a result it follows from the definition relation (23*) that the corresponding expression of the local neo-Bernoullian index is according to Machina's theory

$$(20) \quad \pi^*(x; p) = \sum_k (\partial G^* / \partial \mathcal{M}_k) f^k(x) \quad (43) \text{ (ou A 22)" (M. 358-359)}$$

"The transformation $y = f(x)$ is a pure mathematical transformation devoid of any economic or psychological meaning". (M. 378)

4.3 The incompatibility of the two expressions π and π^* of the 'local utility' function over the whole range $0 \leq x \leq M$. Impossibility Theorem

"Thus the two calculations lead to two expressions π and π^* of "local utility". According to Machina they should be identical up to within a linear transformation for any value of x ($0 \leq x \leq M$)

$$(21) \quad \pi^*(x; p) = \lambda \pi(x; p) + \mu \quad (\text{A. 23})$$

in which λ and μ are functions of the M_k (or \mathcal{M}_k) independent of x.

"Thus, we should have,

$$(22) \quad \sum_k \frac{\partial G^*}{\partial M_k} f^k(x) = \lambda \sum_k \frac{\partial G}{\partial M_k} x^k + \mu \quad (1 \leq k \leq 2n-1) \quad (A.24)$$

whatever the value of x over the whole range $(0, M)$ and whatever the increasing function $f(x)$.

"As the partial derivatives $\partial G / \partial M_k$ and $\partial G^* / \partial M_k$ and the coefficients λ and μ depend only on the M_k , and are *independent of x* , the relation (24) is *totally impossible* since there is an infinity of values of x over the range $(0, M)$. In other words it is impossible to find any functions λ and μ of the M_k , independent of x , such that conditions (24) could be valid for any value of x and for any function $f(x)$ " (Appendix A, M. 381) ¹⁵.

"This *Impossibility Theorem* shows the *impossibility of simultaneously meeting the three following conditions* : - *definition of the local neo-Bernoullian index in the discrete case* from relation (23*); - *its validity over the whole interval $(0, M)$* (§ 1.1.2 and 1.3.1); and *its definition up to within a linear transformation* (§ 1.1.3 and 1.3.1).

"Thus, the validity of the local neo-Bernoullian index definition is *necessarily limited* to the n considered values of x_i in the case of a discrete distribution of order n ; as a result *no derivative of any order of the local index can be calculated*. Consequently the coefficient ρ of risk aversion cannot be determined, *the Hypothesis I and II cannot be explicated, and the very foundation of Machina's theory does collapse in the discrete case*.

"This *Impossibility Theorem* is all the more significant as all the empirical examples presented by Machina as supporting the local utility theory relate to discrete distributions.

"Till now Machina has been *totally unable* to present any valid objection to the demonstration of the *Impossibility Theorem* ⁸ (M. 359).

"n.8 : In his answer of April 10, 1986, to my two memoirs (1986a) and (1986b) sent on March 18, Machina states that my proof of the impossibility theorem was erroneous, but, as I have indicated in my two memoirs (1986d) and (1986c) sent on April 18, his objections were based on mathematical mistakes. Machina has not been able to raise any objection to these two memoirs, and also to my synthesis memoir (1986i) presented at the Aix Conference.

During the debate of June 12, 1986, Machina again did not present any objection to the impossibility theorem. He merely made some comments on his 1986 memoir (relating to some points of his 1982 and 1983a memoirs), which in no way answers the criticisms presented in my five memoirs (1986a), (1986b), (1986c), (1986d) and (1986i), and which in any case, makes no reference to these five memoirs" (M. 392-393) ¹⁶.

As a matter of fact *the equality of the two expressions (6) and (18) of G and G^* (§ 2.d and 4.2 above) represents relatively to x a mathematical identity which can be used in any calculation.*

4.4 Another proof of the Impossibility Theorem

"Another proof of the *Impossibility Theorem* can be deduced from the general theorem on the determination of a discrete distribution as a function of $2n-1$ consecutive moments (§ b above) by considering the following two expressions for ϕ

$$(23) \quad \phi = G (M_q, \dots, M_{q+k}, \dots, M_{q+2n-2}) \quad (P.3)$$

$$(24) \quad \phi = G^* (M_r, \dots, M_{r+k}, \dots, M_{r+2n-2}) \quad (r > q) \quad (P.4)$$

with calculations similar to those of Appendix A (Allais, 1986b, Appendix, p. 15-16)." (*Postscript*, M. 401) ¹⁷.

As I have stressed in the Annex of my 1986b memoir (p. 15), "this another proof is somewhat simpler than the preceding one, and it has the essential advantage to be independent of any transformation $y = f(x)$ ".

4.5 Some Complementary Remarks

A few essential points result from the preceding considerations :

- 1- In the discrete case, there exists an infinity of functions $\pi (x ; p)$ which satisfy the *definition relation* of the local utility (relation 3 of § 2.b above).
- 2- All these functions are identical up to a linear transformation for the x_i values of x .
- 3- However, they are not identical up to a linear transformation for any value of x .
- 4- Therefore, there is no local utility function which would be well defined and differentiable over the entire interval (O, M) , since on the discrete case every local utility function is only defined for the values x_i of x .
- 5- In view of the properties stated by Machina himself (Sections 2 and 3 above) of local utility, the proof above of the *Impossibility Theorem* is crystal-clear and indeed devoid of any defect.

4.6 *The case of continuous or mixed distributions*

As I stressed in § P 1.3.e of my 1987 *Postscript* (p. 401) :

"e.- Finally the *Impossibility Theorem* is valid only for *entirely discrete* distributions, since in the case of continuous or mixed distributions, it is impossible to express the preference functional ϕ as a well defined function of the infinite series of moments $M_1, M_2, \dots, M_k, \dots$ (note 34 above)" (P.1.3.e, M.401).

A striking example of Machina's mistakes

5.- After I sent to Mark Machina my March 18, 1986, letter and my two memoirs joined to it, namely the 1986a one, "*On the Determination of a Discrete Distribution of Order n from 2n-1 Successive Moments*", and the 1986b one, "*Impossibility of the Validity of Machina's Function of Local Utility in Any Finite Domain*", Machina sent immediately to me an answer of April 10, 1986, in which he underscored that my memoir was essentially resting on errors. He wrote in particular :

"Unfortunately the main results in this paper are incorrect. Let me first give a formal demonstration of this, and then point out the errors in your arguments. I will consider each argument in turn".

5.1 - In fact his letter rested essentially on a refutation of relation (22) of § 4.3 above which was grounded on the consideration of the derivations with respect of the p_i of some relations of my 1986b memoir. *These derivations had been made by taking the p_i as independent variables* and by identifying term by term the developments of the two expressions above of $d_p\phi$ (relations 10 and 19 of § 2.e and 4.2 above). Then he showed easily that one must have $\lambda = 1, \mu = 0$.

As I did point out to Machina in my 1986c memoir, "*Machina's Local Utility and the Allais Impossibility Theorem. Additional Comments*" (20 p.), enclosed in my letter of April 18, 1986, that deduction was *completely erroneous* and rested in fact on an *incredible misunderstanding* of the principles of elementary differential calculus :

"The brutal identification of the coefficients of the dp_i in the two expressions of $d\phi_p$ would lead to the conclusion that we would have $\lambda = 1, \mu = 0$.

However such a deduction would be in general completely wrong. In fact the coefficients of two differential developments of a same function could only be identified if the corresponding variables would be independent. Such is not the case since the sum of the p_i is equal to one.

"All what can be said, replacing for instance dp_n by its expression

$$dp_n = - [dp_1 + \dots + dp_{n-1}] \tag{43}$$

is that we have

$$\sum_k \frac{\partial G^*}{\partial M_k} [f^k(x_i) - f^k(x_n)] = \sum_k \frac{\partial G}{\partial M_k} [x_i^k - x_n^k] \quad (1 \leq i \leq n-1) \tag{44}$$

relation which is obtained by identifying the coefficients of the differentials dp_i of the $n-1$ independent variables p_1, p_2, \dots, p_{n-1} in the two expressions of $d_p\phi$ ".

What to say here, if not that mathematics is a wonderful tool, but one has to know how to use it ^{17*}.

5.2- I completed in details, in my 1986d memoir, "*Machina's Fundamental Wrong Statements on the Allais Impossibility Theorem*" (12 p.), enclosed also in my letter of April 18, 1986, my analysis of the numerous other mistakes made in three pages only of Machina's letter of April 10, 1986, and I pointed out in my concluding remarks (p. 7-8) :

"As a matter of fact Machina's letter *does not respond in any way* to the two Allais memoirs (1986a) and (1986b). It eludes the real questions, and the presented arguments, absolutely wrong, boil down to throw a smoke-screen on what is essential.

What is essential is :

- a - the Allais statements concerning the moments of a distribution (Allais, 1986a).
- b - the two expressions of π and π^* of the local utility function deduced *according to Machina's definition* from the two expressions of the preference function ϕ .
- c - *the Machina statement* according to which if two calculations lead to two different expressions π and π^* of the local utility function, they should be tied by a linear relation in which the coefficients λ and μ depend only of the x_j, p_j and are independent of x .
- d - *the Machina statement* according to which the local utility function is valid over the whole interval $[O, M]$.

"In the Allais memoir on the *Impossibility Theorem* it is absolutely necessary to *distinguish very carefully*.

A - the Allais statements which correspond to Allais theory like the statements (a) above.

B - the statements which are immediate consequences of Machina's theory like those corresponding to the Machina's statements as the statements (b), (c) and (d) above.

"Instead of deviating the discussion from what is essential, and stating wrong arguments, Machina *should precise explicitly* the propositions with which he is in agreement, and those with which he does not agree, and why".

5.3 - In my letter of April 18, 1986, sending my two 1986c and 1986d memoirs in answer to the Machina's letter of April 10, 1986, I wrote especially :

"In fact I cannot but ask myself if you *really* understand the meaning and the implications of your formulation of the local utility, and what is *really* in question.

"Indeed your answer is altogether superficial. It does not examine what is fundamental and does not answer to the real questions.

"Thus it seems to me necessary to request you very firmly to answer *explicitly and really*, and that *point by point*, to my *two* papers 1986a and 1986 b ..."

"I think that my two papers 1986a and 1986b are sufficiently clear, but considering the *unbelievable* mistakes of interpretation shown by your letter, I have written a new paper 1986c underlining again the main points of my paper on the impossibility theorem.

"In addition you will find in a fourth paper 1986d the answers to the comments of your letter of April 10, 1986".

"I think that our next Round Table at Aix en Provence can be *extremely useful* and *suggestive* for all people and for both of us, but for that it is evidently *absolutely necessary* that it be prepared *with the greatest care and the greatest clarity*.

"No efficient exchange of view can take place if you don't compel you to answer *point by point* to the arguments presented in my two memoirs of March 18, 1986a and 1986b, and my two complementary papers of April 19 and 20, 1986c and 1986d. From the scientific point of view, that is the only conceivable method".

And I added

"*In any case it would be without any doubt eminently desirable that before hastily concluding that the others are making mistakes, you should begin by seriously asking yourself if you are not yourself into error*".

Notwithstanding my pressing and reiterated requests, by telegrams and letters of April 18, May 6 and May 21, *I could neither obtain any answer from Machina to my letter of April 18, 1986, nor any observations on my memoirs 1986a, 1986b, 1986c, and 1986d*. In fact, I only received a telegram of May 7, 1986, stating : "*I will bring a written response to the Conference*". But that response was not brought to the Aix Conference.

In his letter of June 3, 1986, Machina simply sent to me his communication to the Aix-en-Provence Conference, "*Cardinal Properties of Local Utility Functions*", which did not answer any of my questions and which, in its "*References*" section, did neither mention any of my memoirs on local utility, nor my 1983 memoir, "*The Foundations of the Theory of Utility and Risk*", nor any of my four memoirs 1986a, 1986b, 1986c, and 1986d pertaining to the Impossibility Theorem. *Particularly significant* is the absence of any reference to my 1986a memoir, "*On the Determination of a Discrete Distribution of Order n from $2n-1$ Successive Moments*" (March 18, 1986, 15 p.) and to my 1986d memoir, "*Machina's Fundamentally wrong Statements and the Allais Impossibility Theorem*" (April 20, 1986, 12 p.)

The Aix-en-Provence Conference (10-13 juin 1986)

6.- I presented to the Aix-en-Provence Conference of June 1986 a memoir 1986i, "*Critical Analysis of the Machina Local Utility Theory. Working Paper for the Machina-Allais Debate*" (28 p.).

6.1 - *On June 12, 1986, during the presentation of my Working Paper and the ensuing debate, I posted four large posters reproducing the main formulas of my Working Paper. One of these posters, Poster Nr. II, showed the basic formulae of the proof of the Impossibility Theorem in the discrete case. They were essentially relations (20*), (31), (32), (37), (38), (39), (40), (23*), (42 or A 20), (43 or A 22), (A 23) of my 1986 memoir, published in 1988 (all these relations are reproduced above).*

6.2 - In my 1987 *Postscript*, I precisely pointed out :

"During the Debate of June 12th, 1986, while contesting relation (23) of Appendix A for reasons he left unexplained, Machina stated that he agreed fully with relations (20) and (22) of this Appendix" (*Postscript*, p. 400).

I recall here that these three relations were the following :

$$(25) \quad \pi(x; p) = \sum_k \frac{\partial G}{\partial M_k} (M_1, \dots, M_k, \dots, M_{2n-1}) x^k \quad (1 \leq k \leq 2n-1) \quad (\text{A.20 ou 42})$$

$$(26) \quad \pi^*(x; p) = \sum_k \frac{\partial G^*}{\partial \mathcal{M}_k} (\mathcal{M}_1, \dots, \mathcal{M}_k, \dots, \mathcal{M}_{2n-1}) f^k(x) \quad (\text{A.22 ou 43})$$

$$(27) \quad \pi^*(x; p) = \lambda \pi(x; p) + \mu \quad (\text{A 23})$$

This agreement of Machina on relations (A 20) and (A 22) is of a decisive importance in view of his 1994 memoir. It was in fact a mere confirmation of his previous agreement in his letters of January 19, 1984, January 17 and February 17, 1986¹⁸. That agreement was given during the Plenary session of June 12, 1986, which more than 150 participants attended¹⁹. This debate has been attended by all participants with an intense and passionate attention.

6.3- The questioning by Machina of relation (A 23) has been all the stranger, as, in the same Conference, he presented a paper entitled *Cardinal Properties of "Local Utility Function"*, p. 339-344 of the Volume edited by Bertrand Munier in 1988.

If, indeed, local utility is to be characterized at the same time by an "*Additive Invariance*" and by a "*Multiplicative Invariance*", that means that it is only defined up to a linear transformation. As however, according to Machina, "*the local index $\pi(x; p)$ is well defined on the whole range $0 \leq x \leq M$* ", it follows that if two different calculations lead to two different expressions $\pi(x; p)$ and $\pi^*(x; p)$ of the local utility function, these two determinations are necessarily linked by a linear relation, which means that they verify relation (A 23) above for any value of x (see also note 13 of § 3.c and § 3.e.3 above).

6.4- In fact, as the final text of the *Postscript* of April 27, 1987, was sent to him by Bertrand Munier, Machina had any possibility to object to my writing of the *Postscript* with respect to the debate of June 12, 1986, if it did not give a faithful account of the debate. *The absence of any objection from his part can therefore be regarded as a confirmation.*

I think I should also add that, during that debate, *no single participant* has ever questioned the general proof of the *Impossibility Theorem*, and *especially* the three relations above (A 20), (A 22), and (A 23).

6.5- It seems necessary to recall why I had been led to write the *Postscript*. The best to that effect is to reproduce here its two first paragraphs :

" I thought that my proof of the *Impossibility Theorem* showing the non-existence in the discrete case of a local utility function, continuous and differentiable over the whole interval $(0, M)$, would be sufficient to induce Machina to rectify the mathematical analysis presented in his *Econometrica* paper and that he would specify its basic condition of validity, namely the absence of any discontinuity in the cumulative probability function.

Since Machina has not presented any observation whatsoever on my paper on the local utility theory, especially on the *Impossibility Theorem* and its implications, and since it seemed impossible to leave the reader in the dark regarding the invalidity of the proof presented as general by Machina, the editor of this volume, Bertrand Munier, asked me to specify the points on which Machina's mathematical theory is effectively not right, since its implications are in contradiction with my *Impossibility Theorem* relating to the discrete case ".(P. M. 398).

The repeated refusals to answer of Machina

7.- To make clear for the reader that Machina *repeatedly* refused to answer the questions which I kept asking him *before* and *after* the Aix-en-Provence Conference, it seems necessary to reproduce here some points of my letters to Machina.

7.1 My letter of May 21, 1986

"The least which can be said is that very few people understand your theory. Consequently it is evident that our Round Table could not be efficient if not very carefully prepared ...

"There is no difficulty at all for you to answer to the observations I have presented in *my four papers (1986a), (1986b), (1986c), (1986d), and in my letter of April 18, 1986 ...*

"I have prepared very carefully my two new papers 1986c and 1986d, and I have lost some time for that. It is absolutely necessary that you present your comments on these two papers without delay and in any case before the Round Table.

"*Nobody could understand that you would not submit to this scientific requirement ...*

"What is really interesting for the Conference is not to know who is right or wrong, but to have a clear idea of the scientific points which are in question ...

"Here personal questions are absolutely secondary. The interest of science alone should prevail.

"In any case any passionate element must be absolutely eliminated. I think you will agree completely with this requirement and the necessity to preserve our friendly relations".

7.2 The letter from Machina of July 8, 1986

Following my letter of June 17, 1986, in which I had asked *again and in a very pressing manner* Machina to tell me what his observations on my two 1986c and 1986d memoirs and on my 1986i memoir were, *before I started to write the final version of my memoir of November 8, 1986, "A New Neo-Bernoullian Theory : The Machina Theory. A Critical Analysis"*, Machina let me definitely know that he declined to answer :

" In light of your recent letters, I feel that we have probably reached a point where neither of us believes that the other has adequately responded to the relevant points we have each raised, and that further attempts to do so via letters or discussion would probably be fruitless. Accordingly, I feel it is best that each of us try to present our respective views on the existence and cardinal properties of local utility functions in our separate and independent writings on these issues, and let readers in the profession decide for themselves whose mathematics is correct, which are the relevant issues, and how well each of us has handled them ".

7.3 *My letter of August 4, 1986*

As an answer to his July 8, 1986, letter, I sent Machina a new letter, to ask him to reconsider his decision, as follows :

"I wish that our friendly relations could be preserved and that in no case they could be jeopardized by any polemic climate which would be absolutely regrettable.

"For these reasons I cannot agree with the point of view you express in your letter of July 8th ...

"You say that the best would be "to let the readers decide for themselves *which are the relevant issues*". But how could they do that if they don't know the *precise points* on which you disagree with the objections I have presented ? ...

"As my Aix memoir properly revised and completed will be published in any case, nobody could understand that you would have refused to precise in due time the propositions with which you agree and those with which you consider as erroneous, especially as regards the discrete case analysed in Appendix I ...

"As far as I am concerned I have never refused to anybody to present my observations on the criticisms which were presented. On the contrary I have always considered these requests as very useful. And to my knowledge there is *no example in the whole literature* of any author who refused to answer to the criticisms presented against his works.

"Personally I will in no case take the responsibility of any breaking of our friendly relations.

"But you should realize that your refusal to present to me personally your observations (*before any publication from you or from myself*) on my above mentioned memoirs on your theory (the writing of which as implied for me a quite important time) could only appear to me as a discourteous attitude, the only meaning of which for me would be a deliberate breaking by you of the friendly relations we enjoyed till now, and in this case naturally I would only have to derive the consequences of this breaking".

In his response of August 27, 1986, Machina simply referred me to his *Econometrica* paper of March 1982. *This paper does not deal in any way with the particular case of discrete distributions. Its analysis refers only to mixed distributions* ²⁰.

7.4 *The 1994 Memoir of Machina*

One can only wonder indeed, after eight years during which he remained silent, at the reasons why Machina was led to write his 1994 memoir. Maybe he thinks to have found today the elements of an appropriate and well-grounded response to my objections of 1986. If such is the case, it is only easy to show that he is making again a very serious mistake in view of his previous statements.

The alleged 1994 refutation by Machina of the Impossibility Theorem

8.- The preceding remarks allow me to easily show in what the 1994 paper by Machina rests on erroneous foundations in view of his previous statements as expressed in his correspondence.

8.1 Machina's presentation of Allais' proof

1 - Machina writes in his *Section II* (commentaries on relation 16 and 17)

"To calculate the local utility function $U(.,P)$ of the preference functional $H(.,)$, Allais takes the differential expansion of its moment representation function $G(M_1, \dots, M_{2n-1})$, to obtain (19), so that (according to Allais) the local utility function of $H(.,)$ for outcome values $x \in [0, M]$ must be (20)".

In fact, Allais' effective text (M. 380) reads ²¹

" In fact as according to (4) and (6), we have

$$d_p \phi = \sum_i \left[\sum_k \frac{\partial G}{\partial M_k} (M_1, \dots, M_k, \dots, M_{2n-1}) \frac{\partial M_k}{\partial p_i} \right] dp_i \quad (A 19)$$

$$= \sum_i \left[\sum_k \frac{\partial G}{\partial M_k} (M_1, \dots, M_k, \dots, M_{2n-1}) x_i^k \right] dp_i$$

in which $d_p \phi$ represents the differential of ϕ in relation to the variables p_i , and according to Machina's definition of local utility and relation (46), we find

$$\pi(x; p) = \sum_k \frac{\partial G}{\partial M_k} (M_1, \dots, M_k, \dots, M_{2n-1}) x^k \quad (1 \leq k \leq 2n-1) \quad (A 20)''$$

2- Machina similarly writes in his *Section II* under his relation (17)

" From equations (14) and (15), a similar differential expansion of $G^*(\mathcal{M}_1, \dots, \mathcal{M}_{2n-1})$ yields (21) so that (according to Allais) the local utility function of $H^*(.)$ for outcomes $x \in [0, M]$ must be (22) ".

In fact, Allais' effective text (M. 381) reads ²¹

" Similarly, according to (17) and (18) we have

$$d_p \phi = \sum_i \left[\sum_k \frac{\partial G^*}{\partial \mathcal{M}_k} (\mathcal{M}_1, \dots, \mathcal{M}_k, \dots, \mathcal{M}_{2n-1}) f^k(x_i) \right] dp_i \quad (A 21)$$

and consequently according to Machina's theory

$$\pi^*(x; p) = \sum_k \frac{\partial G^*}{\partial \mathfrak{M}_k} (\mathfrak{M}_1, \dots, \mathfrak{M}_k, \dots, \mathfrak{M}_{2n-1}) f^k(x) \quad (\text{A } 22)$$

in which the partial derivatives $\partial G^*/\partial \mathfrak{M}_k$ only depend on the \mathfrak{M}_k and are independent of x ".

3 - It can be seen from these two quotations that Machina has *deliberately* replaced *twice* the underlined expressions "*according to Machina*" of my text by the expressions "*according to Allais*". *By this very replacement he falsifies indeed my text and deliberately misleads the reader.*

In fact, I inquired of Machina and made sure, at the time, that in the discrete case the definition relation of local utility is effectively the following one ²²

$$(28) \quad d_p \phi = \sum_k \pi(x_i; p) dp_i \quad (23^*)$$

Second, *I also asked Machina and made sure, at the time, that given (23*) and (A 21) expression (A 22) effectively corresponds to the expression of local utility according to Machina* ²³.

Hence indeed we can deduce from relations (23*) and (A 21) that expression (A 22) above *represents effectively another determination of local utility according to Machina* (§ 3.e.3 above).

It is therefore *really unduly misleading* from Machina to deliberately replace, twice, the expressions "*according to Machina*" by the expressions "*according to Allais*". It is indeed purely and simply a *falsification of my text* which is morally and ethically absolutely inadmissible.

8.2 The alleged first Allais' error

1 - Contrary to Machina's assertions in *Section III of his paper* the real question is : "*If, by two different calculations we obtain two different determinations of local utility, $\pi(x; p)$ and $\pi^*(x; p)$, these two determinations must be identical up to a linear transformation*".

Machina asserted himself that property in his letter of January 19, 1984, in response to my letter of January 9, 1984 ²⁴.

2- As I pointed out in *Appendix A* of my 1986 memoir (M. 378) :

"The transformation $y = f(x)$ is a *pure mathematical transformation* devoid of any economic or psychological meaning".

This transformation simply leads to express the preference function ϕ in terms of the moments of the y_i , instead of expressing it as a function of the moments of the x_i . This expression does correspond to a mathematical identity relatively to x_i and it allows to give another expression of the preference function ϕ as a function of x .

3- In any case, the *second proof of the Impossibility Theorem* ²⁵ is *totally independent of any transformation $y = f(x)$* . It rests only on the determination of the preference function ϕ using only $2n-1$ consecutive moments of the x_i . *Machina makes indeed no reference whatsoever to it.*

8.3 The alleged Allais' Second Error

1- According to Machina (*Section II* of his paper) one cannot deduce the determination (A 20) (§ 8.1.1) of local utility from the definition relation (23*) (§ 8.1.3) recalled above ²⁶ and from relation (A 19) above (§ 8.1.1).

2- *But precisely, I have inquired of Machina at the time and made sure that such a deduction is entirely valid for the local utility theory, and he has confirmed on several occasions his complete agreement.*

His agreement was *explicitly expressed* in his letters of January 19, 1984, and of February 17, 1986, and it was *publicly confirmed* in Aix-en-Provence on June 12, 1986 ²⁸.

3- It follows that expressions $\pi(x; p)$ and $\pi^*(x; p)$ of relations (A 20) and (A 22) above (§ 8.1.1 and 8.1.2) *are indeed determinations of local utility in Machina's sense.*

But, as I pointed out in my 1986 memoir, they are only valid for the values x_i of x , and therefore they are not differentiable ²⁹.

In any case it is *very remarkable* that for the values x_i of x the determinations (A 20) and (A 22) of local utility *are effectively linked by a linear relation* ³⁰.

8.4 The crucial relations

1- It follows from the above that *the three crucial relations in the proof of the Impossibility Theorem are the three relations (A 20), (A 22), and (A 23) considered here above* ³¹.

2- *These three relations are based indeed on the agreements, given on several occasions by Machina himself, implying that they do correspond to his theory of local utility* ³². It follows from there that *there is no error whatsoever in the 1986 proof of the Impossibility Theorem*.

3- *If Machina objects today to that proof, he simply and purely puts himself in self-contradiction and he has only got himself to question*.

He cannot in any case object to my 1986 proof of the Impossibility Theorem, which rests on properties on which he explicitly expressed again and again his total agreement ^{32*}.

8.5 Seven significant errors of Machina

Due to lack of space, it is impossible to comment line after line on the mistakes committed by Machina in his 1994 paper ³³. I shall limit myself to comment briefly *seven particularly significant illustrations*.

1- *First Illustration*

According to Machina (*Section III*) it would be impossible to obtain a determination of local utility from a nonlinear transformation $y = f(x)$, which is in any case *purely mathematical*.

This alleged impossibility is wrong, as the illustration given in the Appendix below (§ A 5 and A 6) shows in the case of the expected utility of von Neumann-Morgenstern considered by Machina himself (Sections III and IV of his paper), especially in his relations (25) and (26).

2 - *Second Illustration*

In considering, at the end of *Section III*, "any outcome level y not in (y_1, \dots, y_n) ", Machina considers in fact a *mixed distribution*.

But, as I have constantly stressed, *the Impossibility Theorem is only valid for discrete distributions of a given order n and not for mixed distributions (Postscript, § 1.3.e) (see § 4.6 above).*

In any case, *in the case of a mixed distribution Machina's theory is indeed not correct (see note 20 above).*

Moreover for any discrete distribution of order n , n is given. Therefore :

"The proof of the Impossibility Theorem assumes that all the x_j are different from each other and that no p_j is nil. If one of these conditions is not met, there is degenerescence and the distribution considered is of lower order than n " (*Postscript, M. 400-401*).

3 - *Third Illustration*

In *Section IV "Invalidity of deriving the local utility function from the moment representation function" Machina objects to the possibility of representing identically a random preference function as a function of the corresponding moments, and of calculating its first derivatives from this representation.*

This is an assertion which is quite incorrect. The representation of a random preference function in terms of the moments of its statistical distribution does correspond to a mathematical identity which can be used in any calculation.

Since the first International Conference of 1952 on risk theory, nobody, absolutely nobody, has ever objected to such a representation and its implications ³⁴.

- In fact, it is *always possible* to determine the local utility from the expression of the preference function in terms of its moments. The Appendix hereunder (§ A 4) gives an illustration of this possibility in the case of the neo-Bernoullian formulation of von Neumann-Morgenstern to which Machina does refer in his *Section IV*.

In any case, Machina has himself given in 1986 *his agreement on several occasions regarding the calculation of the local utility from the consideration of its moments* (§ 3.e above).

- Machina's position is all the more untenable as in the discrete case *he never was able to give precise examples of direct calculation of the local utility from the general expression (2) above of the preference function* (§ 2.b).

In fact, and *despite of my reiterated requests*, Machina has been unable to explicit a well defined expression over the whole interval (O, M) of the local utility *in the more general discrete case* of the preference function of order n , $\phi = H(x_1, \dots, x_i, \dots, x_n, p_1, \dots, p_i, \dots, p_n)$ with $\sum p_i = 1$ (§ 2.b), *independently of any consideration of the moments*.

For lack of such an expression and in view of Machina's assertions the Machina local utility theory in the discrete case must be considered as deprived of any real foundation.

This condition is indeed absolutely crucial. A theory which cannot be precised cannot have any scientific value.

• In fact, and as I already wrote in my 1983 memoir, *"The Foundations of Utility and Risk"* (note 102*, p. 98)³⁵:

"If, instead of considering the expression of the preference function f as a function of the moments, one would consider the function

$$\phi = h(g_1, g_2, \dots, g_n, p_1, p_2, \dots, p_n) \tag{1}$$

we would have

$$d_p \phi = \sum_1 \frac{\partial \phi}{\partial p_i} dp_i \tag{2}$$

In this case one does not see how it would be possible to define a "local utility" π , i.e. a function π , meeting a relation similar to relation (39)

$$d_\varphi \phi(\varphi) = \int \pi(x; \varphi) d\varphi(x) dx \tag{39}$$

which could be valid for the discrete case".

• *As a matter of fact, if one would renounce to express the preference function ϕ in terms of its moments, one could generally not determine the local utility as defined by Machina. Without taking into consideration the moments, local utility in the discrete case generally reduces to a myth.*

4 - Fourth Illustration

In his Sections III and IV Machina asserts that he can use von Neumann-Morgenstern's *Expected Utility Theory* as an example invalidating the *Impossibility Theorem*.

In fact, the Impossibility Theorem cannot be applied to any specific case whatsoever of von Neumann-Morgenstern's theory. That theory is indeed a general theory which applies in all cases, whether distributions are continuous, discrete, or mixed. Yet, the Impossibility Theorem is only valid in the case of a discrete distribution considered alone (§ 2.b above), independently of a general preference field which, as the expected utility theory does, would cover all possible cases.

It one considers for instance the "*expected utility*"

$$(29) \quad B = \sum_i p_i u_i$$

with

$$(30) \quad u_i = e^{x_i}$$

the expression of B as a *mathematical expectation of e^x* remains valid for every distribution, whether continuous, discrete or mixed. In such a case, obviously, the *Impossibility Theorem* does not apply ³⁶.

In any case, the e^x function is indeed representable for every value of x by an infinite series, limit of a polynomial of order n , for n infinite ³⁷.

- It is somewhat paradoxical that Machina calls upon *the expected utility theory*, which he *expressly* rejects, to try proving (*altogether in vain*) that the *Impossibility Theorem* is incorrect.

It is proper to stress that the von Neumann-Morgenstern theory which precisely Machina *expressly* rejects cannot be identified with the local utility theory, *which profoundly differs from it*. One could not apply to one what is only valid for the other.

- From a theoretical point of view, there is a very great difference between von Neumann-Morgenstern's theory and Machina's local utility theory. Although the first one has a *great scientific value*, the second one, when considered separately from its *Hypotheses I and II* is only an "*empty box*", according to Robin Pope's expression (my 1986 memoir, § 4.1.3, p. 367), and as far as *Hypotheses I and II* are concerned, *they are devoid of any theoretical or empirical justification* (my 1986 memoir, § 1.2, p. 351 ; § 3, p. 361-364 ; § 4, p. 366-369 ; § 7, p. 375-376).

5 - Fifth Illustration

• If now, in the framework of Machina's theory and in the case of a discrete distribution of order n , one considers the preference function

$$(31) \quad \phi = \sum_i e^{x_i} p_i$$

(moment of order 1 of the e^{x_i} distribution) one has effectively

$$(32) \quad d_p \phi = \sum_i e^{x_i} dp_i$$

so that according to the definition equation (3) of § 2.b the relation (11) of § 2.e above can be written

$$(33) \quad \pi(x_i; p) = e^{x_i}$$

and, in this case, expression e^{x_i} is *effectively representable* by a polynomial of order $2n-1$, *but only for the n values x_i of x .*

In fact, according to § 4.3 and contrary to the 1984 Machina's position (§ 3.e above) the determination

$$(34) \quad \pi(x; p) = e^x$$

of local utility is only valid at the (x_i, p_i) points of the considered distribution³⁸.

*Indeed Machina has never been able to demonstrate that in the discrete case the determination (33) of local utility could be valid for any value of x , and that is precisely the purpose of the Impossibility Theorem to demonstrate that it is not so*³⁹.

6 - *Sixth Illustration*

From his relation (25) of *Section IV* Machina derives his relation (26), and he concludes that local utility cannot be reduced to a polynomial. But this reasoning essentially assumes that relation (26) is valid for every value of x , which is incorrect (see § 4.5.4 above). In fact, the local utility corresponding to a discrete distribution of order n can always be expressed by a polynomial of degree $2n-1$ for all values x_i of the discrete distribution considered (see § 3.e and 8.5.5 above).

- *Indeed* in view of the writing of Machina and in particular of the exclamation mark he uses after relation (26), one can only be amazed. In fact, the impossibility for the e^x function to be represented by a polynomial of degree $2n-1$ for every value of x does constitute precisely the grounding of the *Impossibility Theorem* !

It follows that regarding the interpretation given by Machina of his *Counterexample 2*, one can only wonder whether it rests on an obvious blindness or on a patent bad faith ^{39*}.

7 - *Seventh Illustration*

In his *Section III (Counterexample 2)* Machina writes "To see that something must be wrong with formula 17 ..." (without making really precise what that "something" could well be).

In fact, his formula (17) is *nothing else* than relation (A 20) (§ 6.2 and 8.1.1). With different notations, this relation (A 20) is indeed *nothing else than* relation (47) of my 1984 memoir, explicitly recognized as correct in the framework of the local utility theory by Machina in his letter of February 17, 1986 (§ 3.e above, especially § 3.e.3).

The inconsistency of Machina is here blatant.

8.6 A General Observation

A large part of Machina's considerations rests upon the link he makes between the preference function corresponding to a discrete distribution of order n and a more general preference function, of which the discrete distribution of order n considered in the proof of the *Impossibility Theorem* would only be a component (see for instance § 8.5.2 above).

All these considerations are irrelevant in their substance. To refute a theory which claims to be general, as in the considered case the local utility theory of Machina, one may always limit oneself to the consideration of one particular case.

The particular case I have considered in my proof of the *Impossibility Theorem* is the one of a preference function on which one has only two informations ⁴⁰ :

- a - the value of the preference function ϕ for the values of x_1, x_2, \dots, x_n ; p_1, p_2, \dots, p_n at the considered point (x_i, p_i) with $p_1 + p_2 + \dots + p_n = 1$
- b - the partial derivatives $\partial\phi/\partial p_i$ at the considered point (x_i, p_i) .

In this case, I have shown (and this proof is *rigorous*) that there exists an infinity of local utility functions $\pi(x; p)$ satisfying the *definition relation (3)* of § 2.b above (relation 23* of my 1986 memoir), *but that they are defined only at the considered point (x_i, p_i)* . All these determinations are *identical up to a linear transformation*. The essential conclusion of this analysis is that *there exists no local utility function $\pi(x; p)$ which would be well defined over the whole interval (O, M) and which would satisfy the definition relation (3) of § 2.b above.*

In fact, the local utility in the sense of Machina at the (x_i, p_i) point can only depend upon the informations (a) and (b), and *it is a priori quite obvious that one cannot derive, from the sole informations (a) and (b), a local utility function which would be well defined over the whole interval (O, M) .*

Machina as a Recidivist

9.- The pseudo-refutation of 1994 by Machina of the *Impossibility Theorem* is only a rewriting with a new dressing, but with analogous methods and errors, of his letter of April 10, 1986 ^{40*}.

Fundamentally, beyond numerous basic errors, Machina's approach totally neglects those of my writings which are embarrassing him ; it attributes to me relations which are mere immediate consequences of his theory and on which he previously gave his agreement ; it totally neglects previous statements of his correspondence ; it omits voluntarily every reference to essential memoirs ; it does not correspond to the fundamental principles of intellectual honesty and scientific ethics.

9.1 Non analyzed and omitted justifications

In his 1994 memoir, Machina does not take into account the developments of my 1986 memoir, *the consideration of which shows the erroneous character of his assertions.*

I have given some examples above. Suffice it to recall here some particularly significant illustrations : note (5) ⁴¹ ; note (7) ⁴² ; note (8) ⁴³ ; notes (9) and (10) ⁴⁴ ; note (12) ⁴⁵ ; note (13) ⁴⁶ ; note (32) ⁴⁷ ; note (35) ^{47*} of my 1986 memoir.

The analysis of the texts cited in these notes was in this matter quite fundamental, as especially the analysis of § 3 above shows, but Machina does not make any reference to them.

Likewise, in his References no mention is made of the *Postscript*, although this *Postscript*, written in 1987 on the request of the editor, Bertrand Munier, had a very essential importance in view of the foundations of the demonstration of the *Impossibility Theorem* ⁴⁸.

9.2 *The attribution to Allais of relations implied by Machina's theory and approved by him in his correspondence*

Already in his April 10, 1986, Machina attributed to me on several occasions, as a result of a personal interpretation, relations *resulting in fact from his own theory and from agreements previously given by himself in his correspondence* ⁴⁹.

In his pseudo-refutation of 1994 of the *Impossibility Theorem*, he has recourse again on several occasions to this technique, as I have hereabove given *two especially striking illustrations* ⁵⁰.

In fact, as I had already indicated in *my 1986d memoir of April 20, 1986*, it is essential to distinguish *very clearly and quite explicitly* the relations implied by my own utility and risk theory from the relations resulting from Machina's theory and *explicitly approved by him* ⁵¹.

9.3 *Not taking into account by Machina of his former statements*

Never does Machina take into account in his pseudo-refutation of the *Impossibility Theorem* his former letters, and especially those expressly quoted in my final 1986 memoir, particularly in notes (5), (7), (8) and (13), which did constitute as many *quite essential* justifying references (§ 9.1 above).

9.4 *Not taking into account by Machina of essential memoirs*

My 1986 memoir and its proof of the *Impossibility Theorem* *have been the final outcome* of a long sequence of memoirs, the references of which are indicated in the *Bibliography* which follows my 1986 memoir.

Almost all of these memoirs were written *especially with reference to Machina and none of their results has been published or circulated without that I previously had made sure that they had the agreement of Machina with respect to the interpretation that I had given of his theory, especially in the discrete case.*

Such was *especially* the case of the developments on Machina's theory presented in my 1983 memoir, published in 1984, "*The Foundations of the Theory of Utility and Risk*"⁵². This analysis rests *especially* on the fundamental properties (a), (b), (d), and (e) of § 2 above (§ 3 above). They have been the subject of a whole correspondence with Machina confirming properties (a), (b), (c), (d), and (e) of § 2 above. It is especially in this correspondence, my letter of January 9, 1984, and his answer of January 19, 1984, that, in response to the example which I had sent to him, Machina spontaneously enunciated property (c) of § 2 above.

In fact, as early as 1984, properties (a), (b), (c), (d), and (e) of § 2 above were *totally confirmed* by Machina.

In my 1986a memoir, "*On the Determination of a Discrete Distribution of Order n from $2n-1$ Successive Moments*" I analyzed property (d) in view of the literature (note 34 above), and *Machina has presented no objection whatsoever at the time*.

My proof of the Impossibility Theorem was presented for the first time in english in my 1986b memoir, "*Impossibility of the Validity of Machina's Function of Local Utility in any finite Domain*". As a matter of fact, this proof entirely rests on properties (a), (b), (c), (d), and (e) of § 2 above. *For each one of them, I have relied indeed on the agreements previously given by Machina in his correspondence*⁵³.

It is therefore *absolutely inadmissible* that the references referred to by Machina in his 1994 memoir do not include my 1983, 1986a, 1986b, 1986d, and 1986i memoirs, *all communicated at the time to Machina for observations*, and on which my 1986 memoir rests, especially the proof of my Impossibility Theorem. The same remark holds with respect to the *Postscript* of my 1986 memoir.

9.5 Machina and the intellectual honesty

What to conclude from all these indications ? The least one can say is that intellectual honesty is not a dominating quality of Machina.

In fact, Machina did give again in 1987 two striking illustrations in this respect. As I wrote :

"In two papers, *Choice under Uncertainty : Problems Solved and Unsolved*, (*Economic Perspectives*, 1987, p. 121-154), and *Expected Utility Hypotheses*, (*The New Palgrave, A Dictionary of Economics*, 1987, Vol. 2, p. 232-239), Machina does not cite any of my papers presented at the Conferences at Oslo, Venice, and Aix-en-Provence in which he participated as well.

I believe that although on the scientific plane, one may agree or disagree with certain points of view, it is contrary to scientific deontology to be silent about them whereas quite obviously they bring significant contributions" 54, 55.

I can only regret that Machina could behave here as too many authors do. The fact is that :

"Often too many authors still omit citing my work, maybe when it disturbs them, even when they are perfectly aware with it. For many people incapable of providing a well-founded refutation, their sole weapon is to remain silent. It may not be a very estimable method, but it is often a highly effective one.

"But this is only a rear-guard action : and, sooner or later, the errors of the neo-Bernoullian theories will end up by being clear for all to see. For the last twenty years, in fact, a change of opinion has been making itself more and more manifest, and it continues to grow" 56.

Machina's theory of local utility

10.- On the whole what judgment can one give on Machina's theory ? It seems best to recall here the judgments presented in my 1986 memoir, *A New Neo-Bernoullian Theory : The Machina Theory - A Critical Analysis*, and in its *Postscript*.

10.1 General lack of understanding the real significance of Machina's theory

"The real meaning of Machina's theory is generally not understood. Machina's texts are often elliptic, not very clear, ambiguous, and consequently difficult to read. Moreover they are even sometimes somewhat contradictory ; some very essential propositions are not really justified ; and some do rest on evident mathematical mistakes. This simultaneously explains that so many people do not really understand it, as was very clear at the 1984 Venice and 1986 Aix Conferences, and the many errors of interpretation of some commentators" (M. 347-348).

10.2 The Machina Theory in the Continuous Case

"The local neo-Bernoullian index $\pi(x; \phi)$ cannot be determined directly using experimental data. It can only be deduced from the knowledge of the preference function ϕ . But in this case, any useful information is included in the function ϕ and it is useless to consider the local neo-Bernoullian index ...

In any event there are no psychological or empirical justification of Hypotheses I and II, and they are not testable" (M. 375).

"As a matter of fact, in the case of continuous distributions, the Machina local utility function is no more, under a new name, than the derivative of a line function defined a century ago in 1887 by Volterra" (*Postscript*, § 1.5, p. 402).

10.3 *The Machina Theory in the Discrete Case*

"In the discrete case, it follows from the *Impossibility Theorem* that the local neo-Bernoullian index $\pi(x; p)$ only exists for the values x_j corresponding to the random prospect considered. Consequently, in the discrete case, the local neo-Bernoullian index no longer has derivatives and *Hypotheses I and II can no longer be formulated. Machina's theory is thus deprived of any base*" (M. 375).

10.4 *Overview of the Machina Theory*

"On the whole, the Machina "local utility" theory is *neither well defined nor testable*" (M. 376).

Either one does know the index of preference ϕ and local utility is useless, or one does not know it and local utility cannot be determined. Therefore, either local utility does not exist, or if it exists it is useless.

It follows from these indications that even if my proof of the *Impossibility Theorem* were incorrect (*and it is not*), Machina's theory *would still be devoid of any real scientific value*, and that it is generally quite overvalued. *Indeed it can only delude those who have only a superficial knowledge of it.*

Why then devote so many efforts and so much time to refute such a theory? The explanation is very simple. To write my 1983 memoir, "*The Foundations of the Theory of Utility and Risk*" in which I have tried to give *an objective overview* of seven very significant contributions presented to the Oslo Conference of 1982, I was naturally driven to make up a clear and well-grounded opinion on Machina's theory, a task which, in view of his writings, was not particularly easy.

In fact this critical analysis led me to study thoroughly some very interesting aspects of the theory of utility and risk. As I wrote in my 1994 memoir 57:

"On the whole, the critical analysis I have presented and which is based notably on the theory of moments, *goes very largely beyond the relatively restricted framework of the Machina theory*. As a matter of fact, this analysis has the advantage of focusing attention on some *fundamental aspects* of the theory of random choice, and on the impossibility of a correct representation and a correct explanation of the facts from the "*basic concepts, tools and results*" of neo-Bernoullian theories".

With this reserve, and in view of the reiterated refusals of any useful answer by Machina (§ 7 above), I firmly intended until today not to pursue attempts toward a dialog, the uselessness had become in 1986 more and more blatant. As for me, my 1986 memoir was to be a final point to my analysis of the local utility theory, *which in any case is altogether of secondary importance from a scientific point of view*.

In fact, Machina's 1994 memoir and his pseudo-refutation of the Impossibility Theorem compel me today to answer *and so to loose a precious time*. But Machina should understand that there are limits which one should not cross. As I already put it in my April 18, 1986, letter ⁵⁸:

"In fact I cannot but ask myself if Machina really understands the meaning and the implications of his formulation of the local utility, and what is really in question".

"In any case it would be without any doubt eminently desirable that before hastily concluding that the others are making mistakes, Machina should begin by seriously asking himself if he is not himself into error" ⁵⁹.

APPENDIX

***Illustration of the representation of the preference function
by its moments and of a non linear transformation
in the case of the Von Neumann-Morgenstern
and of a continuous distribution***

A.1 - Preference function considered

Let us consider the continuous case where the preference function is of the form

$$(A 1) \quad \phi = \int e^x \varphi(x) dx$$

where $\varphi(x)$ represents the probability density. Expression (1) is *linear* with respect to the probabilities.

A.2 - Local utility

One has (M. 348-349)

$$(A 2) \quad d_{\varphi} \phi = \int e^x d_{\varphi} \varphi(x) dx$$

From there we deduce (M.355, relation 23) that the expression

$$(A 3) \quad \pi(x; \varphi) = e^x$$

can be considered as a *determination of the local utility*.

A.3 - Expression of the preference function as a function of its moments

One has *whatever* x

$$(A 4) \quad e^x = 1 + \frac{x}{1} + \dots + \frac{x^k}{k!} + \dots$$

and thus

$$(A 5) \quad \phi = 1 + M_1 + \dots + \frac{1}{k!} M_k + \dots$$

with

$$(A 6) \quad M_k = \int x^k \varphi(x) dx$$

A.4 - Second expression of local utility

Thus we have

$$(A 7) \quad d_{\varphi}\phi = \sum_k \frac{d_{\varphi} M_k}{k!} = \sum_k \int \frac{x^k}{k!} d_{\varphi} \varphi(x) dx$$

According to (A 2), expression

$$(A 8) \quad \pi^*(x; \varphi) = \sum_{k=1}^{\infty} \frac{1}{k!} x^k$$

can be equally considered as a determination of local utility.

Thus from (A 4) we have

$$(A 9) \quad \pi^*(x; \varphi) = \pi(x; \varphi) - 1$$

Thus we verify in the continuous case the linear interdependence of π^* and π .

A.5 - Non Linear Transformation

Let us consider the non linear transformation

$$(A 10) \quad y = e^x$$

The probability density $\psi(y)$ of y satisfies the condition

$$(A 11) \quad \psi(y) dy = \varphi(x) dx$$

With the notations used in § 4.2 above, one has thus

$$(A 13) \quad \phi = G^*(\mathcal{M}_1, \dots, \mathcal{M}_k, \dots) = \mathcal{M}_1$$

from which we deduce

$$(A 14) \quad d_{\psi}\phi = d_{\psi}G^* = d_{\psi}\mathcal{M}_1 = \int y d_{\psi} \psi(y) dy$$

and consequently (M. 359) the expression

$$(A\ 15) \quad \pi^{**}(\mathbf{x}; \varphi) = y(\mathbf{x}) + \mu$$

where μ is a constant. π^{**} represents a *third determination* of local utility, and thus according to (A.3) and (A.10) one has the correspondance

$$(A\ 16) \quad \pi^{**}(\mathbf{x}; \varphi) = \pi(\mathbf{x}; \varphi) + \mu$$

A.6 - Commentary

This very simple example allows to verify that in the case of the *expected utility theory* of von Neumann-Morgenstern to which Machina does refer :

- 1) *One can always obtain a determination of the local utility function as a function of the moments of the probability distribution.*
- 2) The expression π^* of the local utility derived from the consideration of the moments is *identical*, up to a linear transformation, to the expression π directly deduced from the consideration of the preference function (when this direct determination is possible, as it is the case of the linear expression (A.1) above).
- 3) After a transformation $y = f(x)$ one can *always* obtain a determination π^{**} of local utility as a function of x from the consideration of the moments of the distribution of y , and that determination is *identical*, up to a linear transformation, to the determination π which can be directly obtained, or to the determination π^* which is obtained from the consideration of the moments of the distribution of x .

I recall that in the case of a continuous distribution, the Impossibility Theorem is not valid (*Postscript* § 1.3.e), (§ 4.6 above).

NOTES

- (1) *Traité d'Economie Pure*, Third edition, 1994, Editions Clément Juglar, 1120 p.
- (2) *Combats pour l'Europe*, 1992-1994, Editions Clément Juglar, 526 p.
- (3) *Economie et Intérêt*, Imprimerie Nationale et Librairie des Publications officielles, Paris, 800 p. 1947.
- (4) *Autoportraits*, Editions Montchrestien, 1989, 150 p.
- (5) Allais, 1995, *Mes Travaux de Physique et l'Anisotropie de l'Espace*.
- (6) *In view of what is in question*, it seems to me *inconceivable* that there could be readers who could not wish to consult *personally* this memoir as Machina does suggest in his introduction : "For the benefit of readers who do not wish to consult Allais 1988".

It is also necessary that the reader read *very carefully* the analysis which I presented in 1983 of the Machina theory in my memoir 'The Foundations of the Theory of Utility and Risk' (p. 42-55 and 97-105).

On this necessity see for instance § 8.1 and § 9 below.

- (7) Since that date, i.e. in the last eight years, I have received no letter or message from Machina on the theory of local utility.
- (8) In the following text I put on the right side the numbers of the relations of my 1986 memoir, with the "A" letter when they come from its Appendix A and with the "P" letter when they come from its *Postscript*. I use quotes for the quotations of my 1986 text. In addition, I add to the quotations of my 1986 memoir, published in 1988, the "M" letter together with the mention of the page. All quotations are presented in smaller characters. I keep naturally the same notations which I used in my 1986 memoir and in its *Postscript*.
- (9) See § 1 above.
- (10) The mentioned relations are those indicated *on the right* of the above relations.
- (11) Property 2 e above (§ 2.e).
- (12) See in particular § 3.e hereunder.

(13) See also Machina's 1982 memoir, n. 43, and his 1986 memoir "*Cardinal Properties of Local Utility Functions*" published in 1988 in *Risk, Decision and Rationality*, edited from the Conference in Aix-en-Provence (p. 339-344).

These papers deal in no way with the case of discrete distributions, but one does not see why what is valid in Machina's view in the case of continuous or mixed distributions would not be valid in the case of discrete distributions.

(14) In the § 2.3.1.a.5, *Illustration of the calculation of the local preference index in a particular case*, of my 1983 memoir (p. 46), the relation (45) is as follows

$$\pi(x; \varphi) = (1 + \lambda \bar{u}) u(x) - (\lambda / 2) u^2(x) \quad (45)$$

It relates to the continuous case. Relation (47) appears in the following § 2.3.1.a.6 under the title : "*Expression of the local preference index in the general discrete case of the Allais formulation*". That formulation (relation 45 of my 1983 memoir) was only a particular illustration.

(15) The relation 21 (or A.23) corresponds to the property (c) of Machina (§ 2.c above, relation 5). On this Machina's property see § 3.c above.

(16) On the answer of April 10, 1986, of Machina to my two 1986a and 1986b memoirs, and on the analysis of his mistakes in my answer of April 18, 1986, and its two 1986c and 1986d annexed memoirs, see § 5 below. On the debate of June 12, 1986, see § 6 below.

(17) As indicated in this text, I gave the first exposition in english of that second proof in the *Appendix* of my 1986b Memoir of March 18, 1986, "*Impossibility of the Validity of Machina's Function of Local Utility in Any Finite Domain*", which was joined to my letter of March 18, 1986, to Machina. No mention whatsoever is made of that 1986b Memoir in the 1994 paper of Machina.

As a matter of fact it seems preferable to replace here in relation (23) above the $2n$ moments M_q, \dots, M_{q+2n-1} of the *Postscript* by the $2n-1$ moments M_q, \dots, M_{q+2n-2} (according to my 1986a memoir) with the addition of the first moment $M_0 = \sum_i p_i = 1$ which is a constant. The same replacement is made in relation (24).

I have given the first proof of the *Impossibility Theorem* in *Appendix III* of my 1984c memoir in french of October 1, 1984, "*Sur la Signification et la Portée de la Théorie de l'Utilité Locale de Mark Machina (On the Meaning and Significance of the Local Utility Theory of Mark Machina)*".

(17*) From relation (44) of my 1986c memoir we deduce

$$(1) \quad \partial G^*/\partial p_i - \partial G/\partial p_i = k$$

where k only depends of the moments of the x_j .

In the same way we have

$$(2) \quad \partial H/\partial p_i - \partial G/\partial p_i = l$$

where H and G are defined by relations (2) (§ 2.b) and (6) (§ 2.d) above and where l only depends of the moments of x_j .

(18) Note (13) of my 1986 memoir (§ 3.0.2 above), of which Machina does not take any account.

(19) *More than 25 of the most important participants have received more than two months in advance my 1986a and 1986b memoirs.*

(20) *In any case and as I pointed out in my Postscript of April 2, 1987 (§ P.1.4., M. 401-402) relation (5) of the 1982 memoir by Machina should be completed in the case of a mixed distribution by three additional terms and it follows that in the most general case, Machina's analysis is not correct.*

(21) It seems necessary to reproduce here relations (A 19) and (A 20) as they were presented in my 1986 memoir. One should write in particular $d_p\phi$ and not $d\phi$ as Machina does (see § 2.b).

(22) See above § 2.b and 3.b.

(23) See above § 3.e.

(24) See § 3.b and § 3.c above. See also the indications in § 6.3 above.

(25) § 4.4 above.

(26) § 2.b, relation (3), § 3.b, and 8.1.3 above.

(27) § 3.e above.

(28) § 3.e and 6.2 above.

(29) § 4.1 above

(30) § 4.1 above.

(31) § 6.2 above.

(32) § 3 above.

(32*) In his letter of September 10, 1984, Machina wrote to me on my analysis of the local utility theory in my 1983 memoir *"The Foundations of the Theory of Utility and Risk"* :

"While I do disagree with many of your remarks regarding the interpretation and especially the empirical properties of my model, your overall description of the formal model is on the whole accurate. Sadly, I must agree with you that many of the participants at the Venice conference indeed did not really understand this model".

As a matter of fact my 1983 analysis did rest especially on properties (a), (b), (d), and (e) (see § 3.b, 3.d, and 3.e above). Here the general judgment of Machina is *especially significant indeed as regards the determination of local utility over the whole interval (O, M) .*

(33) They are too numerous. *There is no single sentence in Sections III and IV of the Machina's paper which does not contain some error or some contradiction with his earlier statements.*

(34) During the 1986 Aix-en-Provence Conference none of the participants ever objected to this representation and its implications.

On this representation see Allais 1986a, *On the Determination of a Discrete Distribution of order n from 2n-1 Successive Moments.*

This memoir rests especially on the following works : 1928, Darmois Georges, *Statistique Mathématique*, Doin, Paris, p. 41-45 ; 1964, Godwin H.J., *Inequalities on Distribution Functions*, Charles Griffin, London, p. 9 and 13-15 ; 1977, Kendall M. and Stuart A. *The Advanced Theory of Statistics*, vol. 1 : *Distribution Theory*, Griffin, Fourth edition, p. 114-115 ; 1943, Shoat, J.A. and Tamarkin, J.P., *The Problem of Moments*, Mathematical Surveys, Number 1, American Mathematical Society, p. 5-6, and 77-78.

(35) On this point see § 2 e above.

(36) See § 4.6 above.

The fifth and sixth illustrations below are related to the fourth, but they are fundamentally distinct in their principles.

(37) See § A.3 below.

(38) § 4.3 above.

(39) See § 4.5 above.

(39*) See § 3.e above and 9.2 below, especially note 49.

(40) As I have stressed in my 1986 memoir (Appendix A.4.a, note 35, p. 383), in the case of the illustration for $n = 2$ of the Impossibility Theorem :

"It is assumed that *no other information is available* relating to the corresponding preference functional ϕ in the case of discrete distributions of order $n > 2$ ".

It was really difficult for me to be more explicit on this *altogether essential point* (§ 2.b above).

(40*) See § 5 above.

(41) § 3.b above.

(42) § 4.1 and 5 above.

(43) § 4.3, 5, and 6 above.

(44) § 2.e and 3.e above.

In his 1994 memoir, in particular in his Bibliography, Machina makes no reference to my 1983 memoir *"The Foundations of the Theory of Utility and Risk"*, of which § 2.3.1 presents a detailed analysis of the local utility theory, and in particular of the local utility concept in the discrete case (p. 42-55, and notes 92-120*, p. 97-105). *That memoir was communicated to him before it was published* (§ 3.b and 3.e above).

(45) § 3.0 and 3.e above.

(46) § 3.0.2. Machina's letters corresponding to n. 13 are analyzed in § 3.b and 3.e above.

- (47) Here is the text of the note (32) of my 1986 memoir 32. On the following proof see Allais 1986a ; — 1986b ; — 1986c ; 1986d ; and — 1986i" (M 394).
Although all these memoirs are altogether essential, none of them is mentioned, except my 1986c memoir.
- (47*) Note 32* (§ 8.4.3) above.
- (48) See in particular § 3.e, 4.4, and 6.2 above.
- (49) § 5.2 above.
 See especially (§ 3.e above) the Machina's letter of February 17, 1986, and my answer of March 18, 1986.
 Indeed, *with an incredible bad faith*, Machina does contest today the interpretation he has explicitly given in 1986 to the local utility concept (§ 3.e.1 above). As I have written to him :
 "What is essential is the renewal of your general agreement of your letter of September 10, 1984, on my calculation of the local utility in the discrete case".
- (50) See § 8.1 above.
- (51) § 5.2 above.
- (52) Section 2.3.1, p. 42-55, and notes 93-120*, p. 97-105.
- (53) § 3 above.
- (54) Allais 1994, *An Outline of any Main Contributions to Risk and Utility Theory*, note 98, p. 216.
 Unfortunately this behavior of Machina is not new. Already in my letter of August 29, 1984, I have been led to write to him :
 "In your paper, *The Economic Theory of Individual Behaviour toward Risk : Theory, Evidence, and new Directions*, 1983, the presentation of my theory is *totally unfair* as regards my 1977 Memoir (Allais and Hagen, p. 437-682).
 "Yet I have given to you in Oslo the list of the contributions of this memoir which you have completely neglected in your review of 1979 in the *Journal of Mathematical Psychology*" (On these contributions see Allais, *Foundations ...*, 1984, note 150, p. 113-114).
 "It is perfectly possible that, like most of the neo-Bernoullians, you don't admit the existence of cardinal utility, yet undeniable, but that does not give to you the right, in a review of the literature which should be objective, to neglect as you do in the last paragraph of p. 104, a general theory using this concept and of which the Neumann-Morgenstern appears simply as a particular case (*Foundations ...*, § 2.2.3, p. 35-42)".
- (55) In her memoir "*Machina's Decision Model : An Empty Box ?*" (University of New South Wales, April 30, 1985) Robin Pope writes :
 "According to Machina, Allais's theory is so inadequate that it did not even warrant a mention in Machina 1982, a 46 pages paper which analyzes in detail the "*Allais Paradox*" and proposals, other than Allais's, for explaining this "Paradox" (See e.g. Machina 1982, p. 278-279, 287-292, 302-306, and 313). This seems a cruel irony when Allais constructed the "*Allais Paradox*" in 1952 as supporting empirical evidence for the Allais theory. The "*Allais Paradox*" has entered mainstream US economists' thinking, but Allais's theory remains taboo, something rarely mentioned, let alone analyzed.
 "Machina's reason for continuing the cold shoulder treatment may be inferred from a review of Allais that he published in a psychology journal. In that review he rebukes Allais for his "sins of omission" (Machina 1981, p. 164). One of the sins was that Allais's "1979 contributions consisted primarily of a restatement of his 1952 views", whereas in the intervening twenty years Allais should have tried to produce new findings that would "operationalize" his models (Machina, 1981, p. 169 and 171). He exhorts Allais : "*Your disciples and intellectual descendants ... are now formalizing your thoughts into well-defined and testable alternative models. Join them !*" (Machina 1981, p. 171).
 The pretentiousness of Machina's last sentence "vaut son pesant d'or" (is worth its weight in gold).
- (56) Allais 1994, id., p. 204

(57) Allais, 1994, id., p. 198.

(58) § 5.3 above.

(59) *Indeed, an excessive pretentiousness associated with a profound misunderstanding of what is in question makes Machina unable to admit his own mistakes. As I wrote to him in my letter of August 4, 1986 :*

"As a matter of fact the whole literature has shown again and again that the best minds are making mistakes, but what is significant is that they recognize their mistakes as soon as the evidence is given to them".

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