Basic Ramsey Model

Assumptions

- There is a benevolent social planner that selects consumption and savings to maximize utility of representative household
- Population (L) grows at rate n
  Note: $L_0 = 1 \Longrightarrow L_t = e^{nt}$, labor is supplied inelastically
- Production function: $Y_t = F(K_t, L_t)$, which has CRS and satisfies the Inada conditions
- Resource constraint: $Y_t = C_t + I_t$
  Assume $\delta = 0$ so $Y_t = C_t + \dot{K}_t$
- Households are dynastic (extended families that care about descendents
  Representative household utility: $U = \int_0^\infty e^{-\rho t} U(C_t) L_t dt$
  Add up across all people: $U = \int_0^\infty e^{-\rho t} U(C_t) e^{nt} dt$
  Utility is increasing in C and concave; $\rho > n$
- Transformed economy (per capita representation): $f(k_t) = c_t + \dot{k}_t + nk_t$ (last term to show need for more capital as population ↑)

Social Planner Problem

$$\max \int_0^\infty e^{-(\rho-n)t} U(c_t) dt$$

$$\text{s.t. } f(k_t) = c_t + \dot{k}_t + nk_t$$
$$k(0) \text{ given, } k_t,\ c_t \geq 0$$

Form the current value Hamiltonian

$$H = U(c_t) + \lambda_t[f(k_t) - nk_t - c_t]$$

FOC

1. $U'(c_t) = \lambda_t$
2. $\dot{\lambda}_t = \lambda_t[\rho - f'(k_t)]$
3. $\lim_{t\to\infty} k_t U'(c_t) - e^{-(\rho-n)t} = 0$

For convergence on balanced growth path, need U to be of constant relative risk aversion (CRRA) form. An example: $U(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$, if $\theta = 1$, $U(c_t) = \ln c_t$.
Note: This form also shows constant elasticity of substitution (CES).

FOC from above:

1. $c_t^{-\theta} = \lambda_t$
   Take time derivative: $\dot{\lambda}_t = -\theta c_t^{-\theta-1}\dot{c}_t$ and plug into (2)
2. $-\theta c_t^{-\theta-1}\dot{c}_t = c_t^{-\theta}[\rho f'(k_t)]$
   Rewrite: $\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(f'(k_t) - \rho)$

This is the Keynes-Ramsey Rule: Inter-temporal analog to the standard effect conditions MRS=MRT.
c increases, stays the same, or decreases depending on whether MPK greater than, equal to, or less than the rate of time preference.

Ramsey Model with Technological Progress

$$\begin{aligned} Y &= F(K, AL) \\ L_t &= L_0 e^{nt} \\ A_t &= A_0 e^{gt}, g = \text{ growth of technology} \end{aligned}$$

Notation for intensive form:

$$\begin{aligned} \hat{y} &= \frac{Y}{AL} \qquad\qquad \hat{k} = \frac{K}{AL} \qquad\qquad \hat{y} = f(\hat{k}) \\ \frac{\partial Y}{\partial K} &= f'(\hat{k}) \\ \frac{\partial Y}{\partial L} &= [f(\hat{k}) - \hat{k}f'(\hat{k})]A \\ \text{Resource constraint: } \frac{Y_t}{A_t L_t} &= \frac{C_t}{A_t L_t} + \frac{\dot{K}_t}{A_t L_t} + \delta\frac{K_t}{A_t L_t} \\ \hat{y}_t &= \hat{c}_t + \frac{1}{AL}\dot{K} + \delta\hat{k}_t \\ \text{Notice: } \dot{\hat{k}}_t &= \frac{d\left(\frac{K_t}{A_t L_t}\right)}{dt} \\ &= \frac{1}{A_t L_t}\frac{dK_t}{dt} - \frac{K_t}{A_t L_t^2}\frac{dL_t}{dt} - \frac{K_t}{A_t^2 L_t}\frac{dA_t}{dt} \\ &= \frac{1}{AL}\dot{K} - \hat{k}(n+g) \end{aligned}$$

Plug back into the resource constraint:

$$\hat{y}_t = \hat{c}_t + \dot{\hat{k}}_t + \hat{k}_t(n + \delta + g)$$

Transform the utility function:

$$\begin{aligned} U &= \int_0^\infty e^{-(\rho-n)t}\frac{c_t^{1-\theta}}{1-\theta}dt, \text{ but notice } \hat{c}=\frac{C}{AL}\Rightarrow c=\hat{c}A \\ &= \int_0^\infty e^{-(\rho-n)t}\frac{\hat{c}_t^{1-\theta}}{1-\theta}A_t^{1-\theta}dt \\ &= \int_0^\infty e^{-(\rho-n-(1-\theta)g)t}\frac{\hat{c}_t^{1-\theta}}{1-\theta}A_0^{1-\theta}dt \end{aligned}$$

Normalize $A_0 = 1$ and require that $\rho > n + (1-\theta)g$

Transformed problem

$$\begin{aligned} \max U &= \int_0^\infty e^{-(\rho-n-(1-\theta)g)t}\frac{\hat{c}_t^{1-\theta}}{1-\theta}A_0^{1-\theta}dt \\ \text{s.t. } f(\hat{k}_t) &= \hat{c}_t+\dot{\hat{k}}_t+\hat{k}_t(n+\delta+g) \end{aligned}$$

FOC

1. $\hat{c}_t^{-\theta} = \lambda_t$
2. $\dot{\lambda}_t = \lambda_t[\rho+\delta+\theta g - f'(\hat{k})]$

The last line comes from:

$$\begin{aligned} \frac{\partial H}{\partial K} &= -[\dot{\lambda}-(\rho-n-(1-\theta)g)\lambda] \\ &= \lambda f'(\hat{k}_t)-\lambda(n+\delta+g) \end{aligned}$$

Note that the entire parenthetical in the e enters as the discount factor.
Take the time derivative of (1) and substitute into (2)

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{1}{\theta}[f'(\hat{k}_t)-\rho-\delta-\theta g]$$

Remaining equilibrium conditions:

$$\begin{aligned} \text{Resource constraint: } f(\hat{k}_t) &= \hat{c}_t+\dot{\hat{k}}_t+(n+\delta+g)\hat{k}_t \\ \text{TVC: } \lim_{t\to\infty}\left(\hat{k}_t\hat{c}_t^{-\theta}e^{-(\rho-n-(1-\theta)g)t}\right) &= 0 \end{aligned}$$

Note that the resource constraint can be rearranged to get the $\dot{c}_t$ and $\dot{k}_t$ equations:

1. $\frac{\dot{\hat{k}}}{\hat{k}} = \frac{f(\hat{k})}{\hat{k}} - \frac{\hat{c}}{\hat{k}} - (g+n+\delta)$
2. $\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{c}}{c} - g = \frac{1}{\theta}[f'(\hat{k})-\rho-\delta-\theta g]$

Decentralized - With Technological Progress

Household Problem

$$\max \int_0^\infty e^{-(\rho-n)t} \frac{c_t^{1-\theta}}{1-\theta}$$

$$\begin{aligned}
\text{s.t. } \dot{\Omega}_t &= r\Omega_t + wL_t + \pi_t - C_t \\
\text{per capita consumption: } c_t &= \frac{C_t}{L_t} \\
\text{per capita assets: } a_t &= \frac{\Omega_t}{L_t} \\
\text{budget constraint: } \dot{a}_t = w + ra_t + \pi_t - c_t - na_t &
\end{aligned}$$

Assumes no Ponzi scheme so that household debt cannot rise faster than r_t:

$$\lim_{t\to\infty} \left(a_t \exp(-\int_0^t [r_v - n] dv) \right) \geq 0$$

Hamiltonian:

$$H = \frac{c_t^{1-\theta}}{1-\theta} + \lambda_t[w + \pi_t + (r-n)a_t - c_t]$$

FOC and TVC

1. $c^{-\theta} = \lambda_t$
2. $\dot{\lambda}_t = \lambda_t[\rho - r]$
   $\frac{\dot{c}_t}{c_t} = \frac{1}{\theta}(r - \rho)$
3. $\lim_{t\to\infty} e^{-(\rho-n)t} c_t^{-\theta} a_t = 0$

Firm Problem

$$\begin{aligned}
Y_t &= F(K_t, AL_t) \\
\text{Note: } \hat{y} &= \frac{Y_t}{AL_t}\ \hat{k} = \frac{K_t}{AL_t}\ \hat{y} = f(\hat{k}_t) \\
\text{Objective function: } \pi &= F(K_t, AL_t) - (r+\delta)K_t - wL_t \\
&= AL[f(\hat{k}_t) - (r+\delta)\hat{k}_t - we^{-gt}]
\end{aligned}$$

FOC

1. $f'(\hat{k}_t) = r + \delta$
2. $[f(\hat{k}_t) - \hat{k}_t f'(\hat{k}_t)]e^{gt} = w$
   Assume CRS$\Longrightarrow \pi = 0$

In the equilibrium economy, plug $a = k$ and $\hat{k} = ke^{-gt}$ into household budget constraint and use the FOCs to get:

$$\dot{\hat{k}}_t = f(\hat{k}) - \hat{c} - (g + n + \delta)\hat{k}_t$$

Also note that: $r = f'(\hat{k}_t) - \delta$ and $\hat{c}_t = c_t e^{-gt}$ to get $\frac{\dot{\hat{c}}}{\hat{c}} = \frac{\dot{c}}{c} - g$

$$\frac{\dot{\hat{c}}_t}{\hat{c}_t} = \frac{1}{\theta}[f'(\hat{k}) - \delta - \rho - \theta g]$$

Note that the budget constraints of the decentralized CE can be written as FOC and resource constraints of the social planner problem. The paths of $\hat{c}$ and $\hat{k}$ are the same.

Ramsey Model with Endogenous Labor

Household Problem

$$\begin{aligned} \max U = \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)N_t, \text{ where } N_t &= \text{population,} \\ l_t &= \text{leisure} \\ \text{time constraint: } l_t + h_t &= \bar{T} \end{aligned}$$

Technology, Preferences, and Constraints

$$\begin{aligned} \text{utility: } U &= \sum_{t=0}^{\infty} \beta^t U(c_t, l_t)N_t \\ \text{production function: } Y_t &= F(K_t, A_t h_t N_t) \\ \text{time constraint: } l_t + h_t &= \bar{T} \\ \text{resource constraint: } C_t + I_t &= Y_t \\ \text{capital accumulation: } K_{t+1} &= (1-\delta)K_t + I_t \\ \text{technological growth: } A_t &= (1+g)^t A_0 \\ \text{labor growth: } N_t &= (1+n)^t N_0 \end{aligned}$$

Transform the economy:

$$\begin{aligned}
\hat{y} &= \frac{Y_t}{A_t N_t} \\
&= F(\hat{k}_t, h_t) \\
h_t &= \bar{T} = 1 - l_t, \text{ if normalize } \bar{T} \\
\hat{y}_t &= \hat{c}_t + \hat{i}_t \\
(1+g)(1+n)\hat{k}_{t+1} &= \hat{i}_t + (1-\delta)\hat{k}_t
\end{aligned}$$

Transform the objective function (more specific utility function adopted):

$$\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\theta}[(c_t v(l_t))^{1-\theta}] N_t \\
&= \sum_{t=0}^{\infty} \beta^t \frac{1}{1-\theta}[(\hat{c}_t v(l_t))^{1-\theta} A_t^{1-\theta}](1+n)^t N_0 \\
&= \sum_{t=0}^{\infty} [\beta(1+n)(1+g)^{1-\theta}]^t \frac{1}{1-\theta}[(\hat{c}_t v(l_t))^{1-\theta}]
\end{aligned}$$

$$\begin{aligned}
b &\equiv \beta(1+n)(1+g)^{1-\theta} \\
U &= \sum_{t=0}^{\infty} b^t \frac{1}{1-\theta}[(\hat{c}_t v(l_t))^{1-\theta}]
\end{aligned}$$

In the above, N_0 and A_0 are normalized to 1.

Social Planner

Using the same utility function and using plug-and-chug:

$$\begin{aligned}
U &= \sum_{t=0}^{\infty} b^t \frac{1}{1-\theta}[(\hat{c}_t v(l_t))^{1-\theta}] \\
&= \sum_{t=0}^{\infty} b^t U(\hat{c}_t, l_t) \\
&= \sum_{t=0}^{\infty} b^t U(F(\hat{k}_t, h_t) + (1-\delta)\hat{k}_t - (1+n)(1+g)\hat{k}_{t+1}, 1 - h_t)
\end{aligned}$$

FOC [define: $\mu \equiv (1+n)(1+g)$]

1. $\frac{\partial U}{\partial h_t} = 0 \Longrightarrow U_c(\hat{c}_t, l_t) F_h(\hat{k}_t, h_t) = U_l(\hat{c}_t, l_t)$

2. $\frac{\partial U}{\partial k_{t+1}} = 0 \Longrightarrow \mu U_c(\hat{c}_t, l_t) = b U_c(\hat{c}_{t+1}, l_{t+1})[F_k(\hat{k}_{t+1}, h_{t+1}) + (1-\delta)]$