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GENERAL EQUILIBRIUM IN A SEGMENTED MARKET ECONOMY
WITH CONVEX TRANSACTION COST:
EXISTENCE, EFFICIENCY, COMMODITY AND FIAT MONEY

BY

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Abstract

This study derives the monetary structure of transactions, the use of commodity or fiat money, endogenously from transaction costs in a segmented market general equilibrium model. Market segmentation means there are separate budget constraints for each transaction: budgets balance in each transaction separately. Transaction costs imply differing bid and ask (selling and buying) prices. The most liquid instruments are those with the lowest proportionate bid/ask spread in equilibrium. Existence of general equilibrium is demonstrated under conventional assumptions, including convexity of the transaction technology. The structure of payment for purchases in equilibrium results from market segmentation and transaction costs. If a lowest-transaction-cost commodity is available, it becomes the common medium of exchange, commodity money. General equilibrium may not be Pareto efficient, but if a zero-transaction-cost instrument is available as the common medium of exchange then the equilibrium allocation is Pareto efficient. Fiat money is characterized as an otherwise worthless government-issued instrument of low transaction cost acceptable in payment of required taxes. Fiat money equilibrium with positively valued fiat money can then be shown to exist.

Keywords: commodity money, fiat money, quasi-equilibrium, general equilibrium, transaction cost

JEL classification: C62, E40, E42

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I. Menger's "Origin of Money" in General Equilibrium

Why do economies use money? is one of the classic issues in the foundations of economic theory, with contributions extending from Smith's *Wealth of Nations*, to the present. *Money*, like *written language* and the *wheel*, is one of the fundamental discoveries of civilization. As an explanation for the use of money, it is not sufficient to say that barter is awkward, requiring a double coincidence of wants. That argument explains why monetary trade is superior to barter exchange. To explain the use of money in a market equilibrium we must also explain how rational optimizing agents choose to use a medium of exchange as part of a market clearing allocation guided by prices.

Despite the evident superiority of monetary trade over barter, there is a counterintuitive --- superficially irrational --- quality to monetary exchange. Monetary trade involves one party to a transaction giving up something desirable (labor, his production, a previous acquisition) for something useless (a fiduciary token or a commonly traded commodity for which he has no immediate use) in the hope of advantageously retrading this latest acquisition. An essential issue at the foundations of monetary theory is to articulate the elementary economic conditions that allow this paradox to be sustained as an individually rational market equilibrium.

Over a century ago, Carl Menger (1892), issued a challenge to monetary theory to explain the apparent irrationality of monetary trade and he proposed a solution to the puzzle based on differing liquidity ('saleability') of commodities:

It is obvious ... that a commodity should be given up by its owner ...for another more useful to him. But that every[one] ... should be ready to exchange his goods for little metal disks apparently useless as such...or for documents representing [them]...is...mysterious...

why...is...economic man ...ready to accept a certain kind of commodity, *even if he does not need it*, ... in exchange for all the goods he has brought to market[?]

[Call] goods ... *more or less saleable*, according to the ... facility with which they can be disposed of ... at current purchasing prices or with less or more diminution...

when any one has brought goods not highly saleable to market, the idea uppermost in his mind is to exchange them, not only for such as he happens to be in need of, but...for other goods...more saleable than his own...By...a mediate exchange, he gains the prospect of accomplishing his purpose more surely and economically than if he had confined himself to direct exchange...Men have been led...without convention, without legal compulsion,...to exchange...their wares...for other goods...more saleable...which ...have ...become generally acceptable media of exchange.

The theory of money Menger proposes here is strongly based on price theory. In Menger's view, not only does a price system provide prices for goods and services; liquidity is priced as well. The price of liquidity ('saleability') is the bid/ask spread, "the ... facility with which they can be disposed of ... at current purchasing prices or with less or more diminution..." A market equilibrium then includes not only a choice of supply and demand; it includes a choice of means of payment. Each buyer and seller chooses the instruments he will accept in exchange for his sales, that he will use to pay for his purchases. He chooses to carry the most liquid ('saleable') goods as carriers of value between his selling and buying transactions. The most liquid goods, those with the narrowest bid/ask spreads, become media of exchange. This happens not because of fiat, legal tender rules, a social contract, or expectations of future acceptability; it occurs because liquidity is clearly and explicitly priced in the spread between buying and selling prices. The price system conveys the information not only of what to buy and sell, but how best to carry value from one transaction to the next. The price system tells traders what good(s) is(are) 'money' simply by conveying the spread between buying and selling prices.

This paper uses the general equilibrium with transactions cost structure of Foley (1970), Hahn (1971) and Starrett (1973) to formalize Menger's (1892) argument. In these

papers transaction costs create a spread between public buying (ask) and public selling (bid) prices in equilibrium. Liquidity (Menger's *saleability*) of an asset is characterized by the bid/ask spread; the most liquid goods are those with a proportionately narrow spread. Liquidity is priced: its price is the bid/ask spread. The concept of a segmented market here is merely the separate budget balance requirement fulfilled at each distinct transaction. Each market segment merely represents a separate transaction: buying food from the supermarket, selling labor to an employer, buying cleaning services from the laundry. Each of these is a separate segment since a separate budget constraint is fulfilled in each one. Markets are segmented (with households fulfilling a separate budget constraint in each segment) reflecting the many distinct transactions a typical household undertakes, generating a demand for a carrier of value between transactions (media of exchange). The most liquid asset, the instrument that provides liquidity at lowest cost, will be chosen as the common medium of exchange. Thus, the choice of a (possibly unique) 'money' is the outcome of optimizing behavior of economic agents in market equilibrium. Menger's views are then fulfilled in equilibrium. The most liquid goods are chosen as the media of exchange.

Allocative efficiency is defined subject to initial endowment and transaction technology; reallocation is necessarily a resource using activity. Nevertheless, general equilibrium need not be Pareto efficient (see Hahn(1971), Starrett(1973), Ostroy and Starr(1990)). However, if there is a transaction-costless common medium of exchange then a general equilibrium will be Pareto efficient (Starrett(1973) and Theorem 3 below).

To prove existence of a general equilibrium in a segmented market with transaction cost this paper combines available treatments. Foley (1970) provides a demonstration of existence of general equilibrium with bid and ask prices and transaction costs in a single unified market. Arrow and Hahn (1971, chapter 6) demonstrates the existence of general equilibrium with externalities. The composite household model below then expands the commodity space and the population of households. Each commodity is treated as distinct depending on which market segment it trades in. Each household is treated as being many distinct counterparts depending on which market segment it trades in. The counterparts are then combined by formalizing an external effect (in the form of a common consumption and common maximand) among them.

The general equilibrium of the composite household model with externalities is then a general equilibrium of the original segmented market economy. The equilibrium concept used is a quasi-equilibrium (compensated equilibrium) rather than a competitive equilibrium. This reflects two technical problems: the complexity of moving budgets among constituent elements of the composite households; the difficulty of assuring ample real income in the presence of a bid/ask spread.

II. Segmented Market Model

Households: There is a finite set of households, H , the typical element is denoted $i \in H$ or $h \in H$.

Goods: There is a finite list of goods, $n = 1, \dots, N$. As in Debreu (1959), the list of commodities is subject to interpretation. They may simply all be spot goods. Alternatively, some may be spot and others for future delivery. Under uncertainty, they may be defined as well by the state of the world in which they are deliverable.

Firms: There is a finite set of firms F , with the typical element $j \in F$. Following Foley (1970), one of the principal activities of a firm is to undertake transactions. Changing the ownership of a commodity --- buying from a seller and selling to a distinct buyer --- is treated like a production activity. It is resource using; the transactions take place at differing prices; the firm undertakes it to make a profit on the difference between buying and selling prices. Hence, the model here interprets the actions of wholesalers, retailers, brokers --- any business whose specialty is making a market --- as a special case of production activity.

Markets: In specifying the structure of markets the present paper breaks with most of the general equilibrium theory, and follows most closely Hahn (1971) and Starrett (1973). The essential elements here are the structure of budget constraints and the evaluation of firm profits. A market in this paper is the locus of transactions. Each household is expected to fulfill a budget constraint on each market separately. Because of the difficulty of assessing discount rates across markets, each firm is supposed to optimize its profits on a single market and confine its transactions to that market. There is a finite set of markets M , each denoted $k \in M$. This construct is used to represent the notion that budgets balance, that payment is made for purchases, at each of many separate

transactions. Thus a multiplicity of budget constraints, one at each transaction, replaces the unified budget constraint of the Arrow-Debreu model. Hahn's (1971) construction of this model emphasized the notion of a sequence economy; that budgets balance at each date. The present treatment is intended to be more agnostic and more general; budgets balance at each transaction.

Prices: Inasmuch as transactions are a resource using activity, undertaken by firms, there will a spread between selling and buying prices. Thus, on any single market, there are two prices for each good. The vector $p^{kS} \in \mathbb{R}_+^N$, represents the vector of selling (bid, wholesale) prices on market k ('selling' as viewed by the public).

Similarly the vector $p^{kB} \in \mathbb{R}_+^N$, represents the vector of buying (ask, retail) prices on market k ('buying' as viewed by the public).

The vector of gross margins, the spread between buying and selling prices on market k is represented by $\pi^k = p^{kB} - p^{kS}$. Note that typically $p^{kB} \geq p^{kS}$ (coordinatewise), and $\pi^k \geq 0$.

Though each firm is active on only one market k , a typical household can transact on a variety of markets. The household must take account of all prevailing prices and price ratios in order to choose the best markets on which to transact. Hence the space of prices facing a household includes two prices for each good in each market. The resulting array of prices lies in $\mathbb{P} \subset \mathbb{R}_+^{2\#MN}$; this is the space of possible price vectors, with typical element $p = (p^{1B}, p^{1S}; \dots; p^{\#MB}, p^{\#MS})$, alternatively denoted $(p^{kB}, p^{kS})_{k \in M} \in \mathbb{P}$.

$x^i \in \mathbb{R}^{2\#MN}$, i 's full transaction plan, $x^{ikB} \in \mathbb{R}_+^N$, $x^{ikS} \in \mathbb{R}_-^N$; note that x^{ikS} is a (negative) vector of sales.

$X^i \subseteq \mathbb{R}^N$, i 's possible net trade space

$u^i(x): X^i \rightarrow \mathbb{R}$, i 's utility function.

$Y^j \subseteq \mathbb{R}^{2N}$, j 's technology set, j is typically active on only one of the segmented markets, $k \in M$, $F(k) \subset F$ is the set of firms j active on market k . The typical element of Y^j is (y^j, y^{jB}) . y^j is j 's net transaction; y^{jB} is the portion of j 's transaction undertaken at the higher retail (ask) prices, p^{kB} . The value of this production plan is $p^{kS}y^j + \pi^k y^{jB}$.

$0 \leq \theta^{ij} \leq 1$, i 's share of firm j

The segmented market model embodies the concept that a typical household will make many separate transactions, with retailers, service providers, an employer, and so forth. In each of these transactions a budget constraint prevails. At prices prevailing in each transaction, budgets must balance; each party delivers value to the other equal to that he receives. Since there is a multiplicity of separate budget constraints, the market is said to be segmented. This concept is formalized as the multiplicity of markets $k \in M$, where M is the set of available distinct markets. In addition, there are transaction costs in each market creating differing bid and ask prices. The notion of transactions as a resource using activity is embodied in firms with a production technology transforming goods between purchased (from the public, 'wholesale' at bid prices) to sold (to the public, 'retail' at ask prices). Finally, transaction costs may differ across markets, so prevailing bid and ask prices may differ as well. Thus with N commodities and $\#M$ market segments there are $2\#MN$ prevailing bid and ask prices. The reason for investigating this construct is to derive the monetary structure that it generates. The multiplicity of budget constraints implies a demand for a carrier of value between transactions. Based on prevailing bid and ask prices, a typical household might then decide to sell good 1 on one segmented market, acquiring there good 2, which it will then take to a second segmented market to trade for its desired purchase, good 3. Trade may occur in this fashion because prevailing transaction costs make it prohibitive to trade good 1 directly for good 3. In this example good 2 acts as a carrier of value from one market segment to another; it becomes a commodity money. The household decision-making that leads the household to choose good 2 to act in this fashion is based on the household's endowment, preferences, and prevailing prices. Menger (1892) argued that the choice of a commodity money will be based on asset liquidity, reflected in the bid/ask spread. Theorem 2 below confirms this viewpoint.

Household i 's actions are $x^i \in \mathbb{R}^{2\#MN}$. $x^i = (x^{i1B}, x^{i1S}, \dots, x^{ikB}, x^{ikS}, \dots, x^{i\#MB}, x^{i\#MS})$. That is, x^i lists i 's N -dimensional buying (a nonnegative vector) and selling (a nonpositive vector) actions on each of the segmented markets $k \in M$. i 's net trade can then be

characterized as $\sigma(x^i) = \sum_{k \in M} (x^{ikB} + x^{ikS})$. $\sigma(x^i)$ is i 's net trade aggregated over the markets $k \in M$.

We characterize prevailing prices as $(p^{kB}, p^{kS})_{k \in M} \in \mathbb{P}$. Then for $x^i \in \mathbb{R}^{2 \# MN}$, we say that $x^i \in D^i(p)$, if x^i maximizes $u^i(\sigma(x^i))$ subject to,

$$p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS} \leq \sum_{j \in F(k)} \theta^{ij} [p^{kS} y^j + \pi^k y^{jB}], \text{ for each } k \in M. \quad (\text{B})$$

Prices $(p^{kB}, p^{kS})_{k \in M}$, $x^i \in \mathbb{R}^{2 \# MN}$, $y^j \in Y^j$ are said to constitute a quasi-equilibrium if $x^i \in \mathbb{R}^{2 \# MN}$ maximizes $u^i(\sigma(x^i))$ subject to,

$$p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS} \leq \sum_{j \in F(k)} \theta^{ij} [p^{kS} y^j + \pi^k y^{jB}], \text{ for each } k \in M$$

or if

x^i minimizes $p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS}$ subject to $u^i(x) \geq u^i(\sigma(x^i))$ for each $k \in M$, and (y^j, y^{jB}) maximizes $p^{kS} y^j + \pi^k y^{jB}$ subject to $(y^j, y^{jB}) \in Y^j$, for each $j \in F(k)$, each $k \in M$,

$$\text{and } \sum_{i \in H} (x^{ikB} + x^{ikS}) - \sum_{j \in F} y^{jk} \leq 0, \text{ co-ordinatewise, for each } k \in M.$$

III. Media of Exchange

Let $p \in \mathbb{P}$, $x^i \in D^i(p)$. i 's net trade consists of $\sum_{k \in M} (x^{ikB} + x^{ikS})$. Let $[\sum_{k \in M} (x^{ikB} + x^{ikS})]_+$

be the N -vector of nonnegative elements in the bracket (zero's in place of other elements), and let $[\sum_{k \in M} (x^{ikB} + x^{ikS})]_-$ be the N -vector of nonpositive elements in the bracket (zero's in

place of other elements). The expression

$$e^i(p) = \sum_{k \in M} (x^{ikB}) - [\sum_{k \in M} (x^{ikB} + x^{ikS})]_+ - [\sum_{k \in M} (x^{ikS})] + [\sum_{k \in M} (x^{ikB} + x^{ikS})]_-$$

represents i 's flow of media of exchange. $e^i(p)$ is gross purchases minus net purchases less gross sales plus net sales; it is the flow of goods in excess of those minimally needed to implement i 's net trade.

IV. Pareto Efficiency

Any reallocation may require incurring transaction costs. The presence of transaction costs and of the wedge between buying and selling prices are not in themselves indications of inefficient allocation. In this literature (Hahn (1971), Starrett (1973)) efficiency is defined relative to initial endowment and transaction technology; necessary transaction costs incurred in moving from endowment to a preferred allocation are a technical necessity and not inefficient. But transaction costs incurred merely in fulfilling budget constraints, condition (B), are regarded as wasted resources. These are the transaction costs incurred in implementing media of exchange, $e^i(p)$. An additional related source of inefficient allocation is preferable reallocations (net of technically necessary transaction costs) discouraged by the prospect of transaction costs incurred in fulfilling budget constraints.¹

V. Assumptions

The following assumptions are familiar in conventional general equilibrium models and correspond essentially to those of Foley (1970). H.1 to H.4 apply to the households of the economy. P.1 to P.4 correspond to the production sector of the economy, including the transactions process as a resource using activity. These are sufficient to develop a model including an equilibrium with a commodity money in Theorems 1, 2 and 3 below. An additional family of assumptions on taxation and fiat money issue, M.1 through M.7, is developed later in the paper to characterize a fiat money equilibrium in Theorems 4 and 5.

H.1 $X^i \subset \mathbb{R}^N$. X^i has a lower bound.

H.2 X^i is closed and convex; $0 \in X^i$.

H.3 $u^i: X^i \rightarrow \mathbb{R}$ is continuous, quasi-concave.

H.4 $x' \gg x^0$ implies $u^i(x') > u^i(x^0)$

P.1 $0 \in Y^j$

P.2 There is no $(y^j, y^{jB}) \in Y^j$ so that $(y^j, y^{jB}) > 0$.

¹ In conversation with Nobuhiro Kiyotaki, he argued that the notion of efficiency above is too restrictive. In the view he expressed --- as I understand it, budget balance, (B), is a technical necessity just as much as is a transaction technology, so the notion of Pareto efficiency should be subject to endowment, transaction technology, and budget balance.

P.3 Y^j is a convex cone.

P.4 $Y = \sum_{j \in F} Y^j$, is closed.

VI. Model of Composite Households with Consumption Externalities

We seek to establish the existence of a general (quasi-) equilibrium in the segmented market model. Rather than prove this directly, we take the approach of restating the model in a way that treats the model as a special case of Foley (1970) with externalities in consumption. That model's sufficient conditions are then adequate to ensure existence of equilibrium in the segmented market model. The strategy of proof is to expand the dimension of the commodity space by a factor of $\#M$, the number of distinct market segments. That is, there are $\#MN$ formally distinct goods. Identical goods in distinct segments are then treated as different goods, with distinct prices, transacted by different firms, and consumed by formally distinct households. In the original segmented market model, each household is active in each of the $\#M$ segmented markets. We now restate this as each household i having $\#M$ distinct counterparts, $ik, k \in M$, active on market segment k . The $\#M$ households are linked in their preferences by an external effect. For each i , and each of the formally distinct households ik' and ik'' (for $k', k'' \in M, k' \neq k''$), ik'' 's consumption plans enter as an external effect in ik' 's utility, as though those consumptions were ik' 's own. Hence we can represent the $\#M$ -segment complex of purchase plans of the typical household i in the original segmented market model, as $\#M$ distinct purchase plans of $\#M$ distinct households linked by an external effect in the composite household model. There are then $\#H\#M$ formally distinct households, each one with preferences linked by an external effect to $\#M-1$ counterparts. Since each household takes fully into account the consumptions of his $\#M-1$ counterparts, and since they share a common utility function, optimization for the complex of $\#M$ distinct households $ik, k \in M$, in the composite household model is equivalent to that of household i in the segmented market model. We demonstrate the existence of equilibrium in the composite household model and then note that the conditions for equilibrium there are precisely equivalent to those of the segmented market model. Hence the segmented market model has a general equilibrium.

In the composite household economy we consider a revised population of households, $HM = \{ik \mid i \in H, k \in M\}$. For each $ik \in HM$, ik 's buying and selling plans are restricted to segment k . $x^{ik} \in \mathbb{R}^{2\#MN}$ with all co-ordinates for markets other than k set identically equal to 0. $x^{ik} = (0, 0, \dots, 0, x^{ikB}, x^{ikS}, 0, \dots, 0, 0)$. Conversely, let $x^{i-k} \in \mathbb{R}^{2\#MN}$ denote the $2\#MN$ -dimensional vector of external effects on ik coming from its counterparts active on markets $k' \neq k$. $x^{i-k} \equiv (x^{i1B}, x^{i1S}, \dots, x^{ik-1B}, x^{ik-1S}, 0, 0, x^{ik+1B}, x^{ik+1S}, \dots, x^{i\#MB}, x^{i\#MS})$ setting at 0 the k -indexed co-ordinates. That is, each household $i \in H$ of the original segmented market model appears in the composite household model as $\#M$ distinct households, one for each segmented market $k \in M$. The separate households are related by an external effect --- each of the separate households appreciates fully the consumption decisions of its counterparts.

Household ik 's utility function is characterized by strong external effects. ik 's preferences are those of i in the segmented market model, applied to ik 's net trades plus those of ik' , $k' \neq k$. That is,

$$u^{ik}(x^{ik}; x^{i-k}) = u^i(x^{ikB} + x^{ikS} + \sigma(x^{i-k}))$$

where x^{i-k} is treated parametrically. The constraint set on ik 's transactions then is X^{ik} defined as

$$X^{ik}(x^{i-k}) \equiv \{ x^{ik} \in \mathbb{R}^{2\#MN} \mid x^{ik} = (0, 0, \dots, 0, x^{ikB}, x^{ikS}, 0, \dots, 0, 0), \\ (x^{ikB} + x^{ikS} + \sigma(x^{i-k})) \in X^i \} .$$

ik 's income, to be spent on market k , comes from sales of goods on k , x^{ikS} , and from ik 's share of profits of firms active on k . We take ik 's shares of firms $j \in F(k)$ to be identical to i 's, θ^{ij} for $j \in F(k)$. Thus ik 's budget constraint is

$$p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS} \leq \sum_{j \in F(k)} \theta^{ij} [p^{kS} y^j + \pi^k y^{jB}] .$$

Prices $(p^{kB}, p^{kS})_{k \in M} \in \mathbb{P}$, $x^{ik} \in \mathbb{R}^{2\#MN}$, $y^j \in Y^j$ are said to constitute a quasi-equilibrium in the composite household model if for each $ik \in HM$

x^{ik} maximizes $u^{ik}(x^{ik}; x^{i-k})$ on $X^{ik}(x^{i-k})$ subject to

$$p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS} \leq \sum_{j \in F(k)} \theta^{ij} [p^{kS} y^j + \pi^k y^{jB}]$$

or if

x^{ik} minimizes $p^{kB} \cdot x^{ikB} + p^{kS} \cdot x^{ikS}$ subject to $u^{ik}(x^{ik}; x^{i-k}) \geq u^i(\sigma(x^i))$ for each $k \in M$, and
 (y^j, y^{jB}) maximizes $p^{kS} y^j + \pi^k y^{jB}$ subject to $(y^j, y^{jB}) \in Y^j$, for each $j \in F(k)$, each $k \in M$, and

$$\sum_{\substack{ik \\ i \in H}} (x^{ikB} + x^{ikS}) - \sum_{j \in F(k)} y^j \leq 0, \text{ co-ordinatewise, for each } k \in M.$$

VII. Results

Lemma 1 (Existence of a quasi-equilibrium in the composite household economy): Assume H.1-H.4, P.1-P.4. Then the composite household economy has a quasi-equilibrium with prices $(p^{*kB}, p^{*kS})|_{k \in M}$.

Proof: Foley (1970), Theorem 4.1. Arrow-Hahn (1971); the discussion on p. 135 demonstrates the existence of a quasi-equilibrium (compensated equilibrium, leading to a competitive equilibrium subject to income conditions, in Theorem 6.2). QED

Theorem 1 (Existence of a quasi-equilibrium): Assume H.1-H.4, P.1-P.4. Then the original economy has a quasi-equilibrium with prices $(p^{*kB}, p^{*kS})|_{k \in M}$.

Proof: Apply Lemma 1. Combine the composite households into original households. For each $i \in H$, the first order conditions to maximize u^{ik} subject to budget constraint in $k \in M$ are identical to those of maximizing u^i subject to budget constraint. QED

Theorem 2 (Demand for media of exchange): Let $p \in \mathbb{P}$, $x^i \in D^i(p)$ and consider $e^i(p)$. Designate some $n^*=1, \dots, N$ and all other $n \neq n^*$, $n=1, \dots, N$. Let $p^{kB}_{n^*} > 0$ and $p^{kS}_{n^*} > 0$, for all $k \in M$. Further, let

$$\frac{p_{n^*}^k}{p_{n^*}^{kB}} = \frac{p_{n^*}^{kB} - p_{n^*}^{kS}}{p_{n^*}^{kB}} < \frac{p_n^{kB} - p_n^{kS}}{p_n^{kB}} = \frac{p_n^k}{p_n^{kB}} \text{ and } \frac{p_{n^*}^k}{p_{n^*}^{kS}} = \frac{p_{n^*}^{kB} - p_{n^*}^{kS}}{p_{n^*}^{kS}} < \frac{p_n^{kB} - p_n^{kS}}{p_n^{kS}} = \frac{p_n^k}{p_n^{kS}} \text{ for all}$$

$k \in M$. That is, on all markets, both on the buying and selling side, good n^* has the narrowest proportionate bid/ask spread of any good. Then the only nonnull co-ordinates of $e^i(p)$ are in good n^* .

Proof: Suppose $e_n^i(p) > 0$ for $n \neq n^*$. Then there is an alternative x^i fulfilling (B) with utility higher than x^i (x^i using more n^* as medium of exchange, less n).

Example 1 (Inefficient allocation in equilibrium): See Starrett (1973) or Ostroy and Starr (1990).

Theorem 3 (Efficiency of allocation with a transaction-costless medium of exchange): Let $p^* \in \mathbb{P}$ be a quasi-equilibrium price vector and $x^{*i} \in D^i(p^*)$ be the corresponding equilibrium trading plans. Let (y^{*j}, y^{*jS}) be the corresponding firm plans. Let there be good n^* so that $p^{*kB}_{n^*} = p^{*kS}_{n^*} > 0$ for all $k \in M$. Then the allocation x^{*i} is Pareto efficient.

Proof: For all i, k, n so that $x^{*ikB}_n \neq 0$ we have $[p^{*kB}_n / p^{*kS}_{n^*}] \leq [p^{*kB}_n / p^{*k'S}_{n^*}]$ for $k' \neq k$. For all i, k, n so that $x^{*ikS}_n \neq 0$ we have $[p^{*kS}_n / p^{*kB}_{n^*}] \geq [p^{*k'S}_n / p^{*kB}_{n^*}]$ for $k' \neq k$.

Denote $r^B_n = [p^{*kB}_n / p^{*kS}_{n^*}]$ for k so that $x^{*ikB}_n \neq 0$. Denote $r^S_n = [p^{*kS}_n / p^{*kB}_{n^*}]$ for k so that $x^{*ikS}_n \neq 0$. Then (r^B, r^S) is a $2N$ -dimensional price vector supporting the allocation $x^{*i}, (y^{*j}, y^{*jS})$.

The allocation is Pareto efficient by the First Fundamental Theorem of Welfare Economics.

VIII. Monetary and Financial Structure

Theorems 1, 2, and 3 above develop the model of commodity money equilibrium. Theorem 1 merely states that the assumptions are sufficient to generate existence of a quasi-equilibrium. Theorem 2 embodies Menger's (1892) argument that commodity money is based on liquidity. Theorem 2 proposes that there be a single good n^* with narrowest proportional bid/ask spread at prevailing prices. Then n^* will be the unique medium of exchange. Clearly Theorem 2 poses a simplified case --- there could be several goods tied for narrowest bid/ask spread or the good with the narrow bid/ask spread could vary across markets. Nevertheless, the underlying principle is clear. Liquidity is priced in the bid/ask spread and the most liquid good(s) will be the medium(a) of exchange.

When a household engages in trade, it may concentrate sales from endowment or concentrate its income from business on one market k' but concentrate its purchases for consumption on another market k'' . The two values, sales (plus profits) and purchases, must balance on each market separately, (B). Hence the household uses a carrier of value, commodity money $e^i(p)$, to shift purchasing power among markets. It seeks to do so in the most advantageous fashion possible, losing as little purchasing power as possible in the process. That is how the household forms its optimizing choice of $e^i(p)$. For arbitrary $p \in \mathbb{P}$, there is no simple general characterization of $e^i(p)$. But Theorem 2 describes the most interesting special case. Suppose --- at prevailing prices --- there is a natural money, n^* , a good with such low transaction costs that the prevailing bid/ask spread makes it the least costly way to move purchasing power across all markets. Then n^* is the only good that will be used as $e^i(p)$. All transactions will either be for directly useful trades --- delivering supplies, fulfilling demands --- or they will be in n^* acting as a medium of exchange.

Jevons (1875) reminds us that money (particularly a commodity money) should possess the following properties: 1. Utility and Value, 2. Portability, 3. Indestructibility, 4. Homogeneity, 5. Divisibility, 6. Stability of value, 7. Cognizability. How do these notions stack up in the current model? Following Theorems 1 and 2, suppose we have $p^* \in \mathbb{P}$, a market equilibrium price vector, with n^* the low transaction-cost instrument in all markets $k \in M$. What can we expect of n^* at p^* with regard to Jevons's specifications?

1. Utility and Value. As stated in Theorem 2, the bid and ask price of n^* must be positive in all markets $k \in M$. For n^* to act as a medium of exchange, it must have a positive price. In order to maintain a positive bid price, n^* must have a desirable use, so the model is fully in agreement with Jevons. Clearly modern fiat money does not directly fulfill this property, so some explanation of its positive value is required. That discussion appears below.

2. Portability, 3. Indestructibility. The resource requirements for dealing in n^* , including its transportation requirements and any n^* used up through deterioration in the process of trade, are included in the transaction/production technology Y^k . The choice of n^* as the low bid/ask spread instrument in equilibrium reflects a low transaction cost.

The resources required for dealing in and transporting n^* , including shrinkage in the quantity of n^* , should be small and of low value. The notion of portability is embodied in Theorem 2 through the transaction technology and its attendant costs.

5. *Divisibility*. Since the commodity space and transaction technologies are assumed to be convex, divisibility is a matter of assumption and definition in this model.

4. *Homogeneity*, 7. *Cognizibility*. A lack of uniformity in quantity or quality of a good will increase the resources required to trade it, since resources will need to be expended in verifying and quantifying the good. Hence the notions of the resource cost and resource saving associated with homogeneity and cognizibility are embodied in the transaction technology and in Theorem 2.

6. *Stability of value*. This issue is trickier to treat; it involves the time and uncertainty structure of the model. Assume a model with a full set of time-dated, state-of-world labeled goods. In an Arrow-Debreu model we'd call this a full set of futures markets under uncertainty. That nomenclature doesn't precisely fit here, since the markets may be available but with high transaction costs so that they are inactive in equilibrium, Hahn (1971). A typical inactive market will have posted bid and ask prices for each future/contingent commodity. But bid/ask spreads may be wide enough so that no trading takes place, so that the risk of price variation is not hedged because hedging itself is too expensive. As events unfold, the price (or more precisely, the array of bid and ask prices) of a commodity may vary as events move down an event tree. The implication of Theorem 2 is that to the extent that 'stability of value' leads to market liquidity (narrow bid/ask spread) then stability is important to the designation of the common medium of exchange, n^* . The link between stability and liquidity is that risk aversion discounts the value of the more unpredictable contingent commodities without reducing their transaction costs, hence resulting in illiquidity.

Example 1 and Theorem 3 emphasize the notion of Pareto efficiency. The example (more precisely, the restatement that the examples of Starrett (1973) or of Ostroy and Starr (1990) still apply) demonstrates that in a segmented market with transaction cost, equilibrium need not be Pareto efficient. However, if there is a medium of exchange that operates with zero transaction cost ('money'), then common general

equilibrium prices can be re-established and the allocation is Pareto efficient by the usual First Fundamental Theorem of Welfare Economics.

Banks, Financial Intermediation

Assume a model with a full set of time-dated, state-of-world labeled goods. Banking (and more complex financial intermediation) functions are just another market making function: buying and selling dated goods (particularly those of the common medium of exchange, n^* , if it exists). Borrowing, or accepting deposits, consists of buying current goods, including n^* . Repaying deposits consists of disbursing corresponding future goods, including future n^* . Lending consists of selling current goods, including n^* while buying future goods. A bank arranging future repayment of a loan is buying future dated goods, including n^* . The present model assumes convexity of transaction technologies, so the intermediation function simply falls on the firms best suited to perform it, those with the lowest transaction costs in the spot and futures markets. However, the scale economy and law of large numbers arguments common in partial equilibrium descriptions of banking cannot be fully accommodated here, since they involve a nonconvexity in the transaction technology.

IX. Fiat Money and Government

The outlines of how to incorporate fiat money can be briefly sketched. There are two issues to be addressed: How does an intrinsically worthless instrument become positively valued in equilibrium? How does this instrument become the common medium of exchange? The answer to the first question is "taxes." The answer to the second is "low transaction costs." Introduce a government in the model with the powers to issue fiat money and to collect taxes. Fiat money is intrinsically worthless; it enters no one's utility function. But the government is uniquely capable of issuing it and of declaring it acceptable in payment of taxes. Adam Smith(1776) notes "A prince, who should enact that a certain proportion of his taxes be paid in a paper money of a certain kind, might thereby give a certain value to this paper money..." (v. I, book II, ch. 2). Abba Lerner(1947) comments

The modern state can make anything it chooses generally acceptable as money and thus establish its value quite apart from any connection, even of the most formal kind, with gold or with backing of any kind. It is true that a simple

declaration that such and such is money will not do, even if backed by the most convincing constitutional evidence of the state's absolute sovereignty. But if the state is willing to accept the proposed money in payment of taxes and other obligations to itself the trick is done. Everyone who has obligations to the state will be willing to accept the pieces of paper with which he can settle the obligations, and all other people will be willing to accept these pieces of paper because they know that the taxpayers, etc., will be willing to accept them in turn.

Taxation --- and fiat money's guaranteed value in payment of taxes --- explains the positive equilibrium value of fiat money². Assume the fiat money also to have very low transaction costs. Then the trick is done. Fiat money becomes the common medium of exchange³. For an example of how this logic works see Starr (2001). To formalize these views we extend the model above by modeling government and fiat money.

Define taxes, fiat money and government in the following way. Government, denoted G , is formalized as another household with distinctive properties for its achievable net trade set, X^G . Tax receipt certificates are good $N-1$. Every household $h \in H$, has a tax quota $\theta^h > 0$, so that there is positive marginal utility from acquiring additional x_{N-1}^h up to the level θ^h . No firm $j \in F$ can produce $N-1$ and no household in H can achieve a net disbursement of $N-1$ (that is, no household is endowed with $N-1$). Government, denoted G , is the unique source of N and is endowed with $N-1$ (more formally, X^G admits the possibility of a net disbursement of $N-1$, tax receipts), but not so much that it becomes a drug on the market. Good N will be treated as fiat money. We assume that no household gets positive utility from good N . Government G , declares its willingness to accept N (which nobody wants) in exchange for $N-1$ (which everybody wants). This amounts merely to defining u^G , G 's utility function, with strictly positive marginal utility for N and for $N-1$. We formalize these notions as⁴

$$M.1 \text{ For each } h \in H, \text{ there is } \theta^h > 0 \text{ so that if } x_{N-1}^h \leq \theta^h, \frac{\partial u^h(x^h)}{\partial x_{N-1}^h} > 0.$$

$$M.2 \ X^G \equiv \mathbb{R}^N - \{(0, 0, \dots, \sum_{h \in H} \theta^h, \sum_{h \in H} \theta^h)\}. \text{ For all } h \in H, \text{ all } x \in X^h, x_{N-1} \geq 0.$$

² See also Li and Wright (1998) and Starr (1974).

³ A more complex argument involves a scale economy, ruled out by the present paper's convexity assumption. If there is a scale economy in transaction costs and if government is a large economic agent, then government transactions in fiat money ensure sufficient scale to result in low transaction costs. Hence fiat money becomes the unique common medium of exchange.

⁴ The partial derivatives representing marginal utilities in the assumptions below are assumed to exist everywhere in the (relative) interior of X^G . The assumptions can be restated without differentiability, but the notion of marginal utility is clearly useful here.

M.3 For all $x \in X^G$, $\frac{\partial u^G(x^G)}{\partial x_{N-1}^G} = \frac{\partial u^G(x^G)}{\partial x_N^G} > 0$.

M.4 For all $h \in H$, $\frac{\partial u^h(x^h)}{\partial x_N^h} = 0$.

M.5 For all $h \in H$, all $x \in X^h$, $x_{N-1}, x_N \geq 0$.

M.6 For all $j \in F$, all $(y^j, y^{jB}) \in Y^j$, $y_{N-1}^j = y_N^j = 0$.

M.7 For all $j \in F$, let $(y^j, y^{jB}) \in Y^j$, let $y^j = y^{''j}$, $y^{jB} = y^{''jB}$ for $n=1,2,\dots,N-2$. Let $y^{''jB}_{N-1} \geq y^{jB}_{N-1}$, $y^{''jB}_N \geq y^{jB}_N$. Then $(y^{''j}, y^{''jB}) \in Y^j$.

Assumptions M.1 through M.7 define the notions of fiat money and taxation. M.1 says that households try to arrange their affairs to pay their taxes. M.2, M.5 and M.6 say that government, G, is the unique source of money, good N, and tax receipt certificates, good N-1. M.3 says that government, G, is willing to accept money, good N, one for one, in exchange for tax receipt certificates, N-1. M.4 says there is no utility to money, good N, for any household; only government G behaves as though money, N, is desirable. M.7 says that tax receipts and money, goods N-1 and N, carry low transaction costs.

Theorem 4 (Existence of a fiat money quasi-equilibrium): Assume H.1-H.4, P.1-P.4, M.1-M.7. Then the economy has a quasi-equilibrium with prices $(p^{*kB}, p^{*kS})_{k \in M}$. Further, $p^{*kB}_N, p^{*kS}_N > 0$.

Proof: Apply Theorem 1 and M.1-M.7. $p^{*kB}_{N-1}, p^{*kS}_{N-1} > 0$ by M.1 and M.7. Then $p^{*kB}_N, p^{*kS}_N > 0$ by arbitrage with M.3. QED

Theorem 5 (Demand for fiat money): Assume H.1-H.4, P.1-P.4, M.1-M.7. Let $(p^{*kB}, p^{*kS})_{k \in M}$ be quasi-equilibrium prices with $p^{*kB}_N, p^{*kS}_N > 0$. Let $x^i \in D^i(p^*)$ and consider $e^i(p^*)$. Then

$$\frac{p_N^{*k}}{p_N^{*kB}} = \frac{p_N^{*kB} - p_N^{*kS}}{p_N^{*kB}} \leq \frac{p_n^{*kB} - p_n^{*kS}}{p_n^{*kB}} = \frac{p_n^{*k}}{p_n^{*kB}} \quad \text{and} \quad \frac{p_N^{*k}}{p_N^{*kS}} = \frac{p_N^{*kB} - p_N^{*kS}}{p_N^{*kS}} \leq \frac{p_n^{*kB} - p_n^{*kS}}{p_n^{*kS}} = \frac{p_n^{*k}}{p_n^{*kS}}$$

for all $k \in M$, $n=1,2,\dots,N-1$. That is, on all markets, both on the buying and selling side, good N has the narrowest proportionate bid/ask spread of any good. Then there is $e^i(p^*)$ so that the only nonnull co-ordinates of $e^i(p^*)$ are in N.

Proof: Apply M.7, Theorem 4 ($p^{*kB}_N, p^{*kS}_N > 0$) and Theorem 2. QED

Theorem 4 merely represents the application of Theorem 1 to the case of the fiat money economy. Positivity of the price of fiat money, that is $p^{*kB}_N, p^{*kS}_N > 0$ for each k , comes from the fiat money structure developed in M.1 through M.7. Good N is desirable since it is desired by G on the same basis as $N-1$ (M.3) and all households want $N-1$ (M.1). These statements hold net of transaction costs since these are small (M.7). Further the scarcity value of N and $N-1$ is ensured by limitations on supply (M.2, M.5, M.6). Theorem 5 merely restates Theorem 2 for fiat money. Since fiat money is a low transaction cost instrument, it is likely to be priced with the lowest bid/ask spread of any good. Then it will be the universal medium of exchange.

X. Conclusion

In an economy with segmented markets and the resultant multiple budget constraints, goods act as carriers of value between transactions. If there are transaction costs, there is an incentive to concentrate this function on the low transaction costs instruments. A price system formalizes this structure in bid and ask prices. Liquidity is priced; its price is the bid/ask spread. There is an incentive to concentrate the medium of exchange function on goods with the narrowest bid/ask spread. This gives the commodity money structure of a segmented market equilibrium (Theorems 1 and 2). Positive value of fiat money issued by government can be supported by fiat money's acceptability in payment of taxes (Theorem 4). Fiat money as the common medium of exchange derives from its low transaction cost (Theorem 5).

Appendix: Foley's transaction cost model and Arrow and Hahn's treatment of external effects

Foley (1970) noted the formal equivalence of the existence of a quasi-equilibrium in Debreu (1962) to its existence in a model with a doubled dimension of the commodity space with a convex transaction technology. Arrow and Hahn (1971) noted that the results demonstrating existence of quasi-equilibrium (compensated equilibrium) could be generalized to a model including continuous external effects among households. The combined result below notes that the same logic means that the Foley (1970) result holds in the presence of continuous external effects among households. All of these results apply in a single unified market. Now consider $\#M$ segmented markets where each household and each firm is allowed to be active in only one segment and each household has ownership shares only of the firms' in its segment. The adapted result below extends the combined result to this segmented market model: continuous external effects are consistent with a quasi-equilibrium (compensated equilibrium) in the segmented market model.

Published Results

(Foley(1970)) Let $\#M=1$, and assume H.1-H.4, P.1-P.4, with no external effects. Then there is a quasi-equilibrium (p^{*kB}, p^{*kS}) .

(Arrow and Hahn (1971)) Let $\#M=1$, and assume H.1-H.4, P.1-P.4, with continuous external effects (each household's utility function is continuous in the consumptions of other households). Let transaction costs be nil. Then there is a compensated equilibrium (quasi-equilibrium) p^* .

Combined Result

Assume H.1-H.4, P.1-P.4, with continuous external effects. Let $\#M=1$. Then there is a quasi-equilibrium (p^{*kB}, p^{*kS}) .

Adapted Result

Assume H.1-H.4, P.1-P.4, with continuous external effects. Let $\#M>1$ but with each firm and household active on only one $k \in M$. Then there is a quasi-equilibrium $(p^{*kB}, p^{*kS})|_{k \in M}$.

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